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On generalized quasi Baer skew monoid rings

Mohammad Habibi[*,‡], Kamal Paykan[†,§]
and Hassan Arianpoor[*,¶]

[*]*Department of Mathematics*
Tafresh University, Tafresh, Iran

[†]*Department of Basic Sciences, Garmsar Branch*
Islamic Azad University, Garmsar, Iran
[‡]*mhabibi@tafreshu.ac.ir; habibi.mohammad2@gmail.com*
[§]*k.paykan@gmail.com*
[¶]*arianpoor@tafreshu.ac.ir*

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In this paper, we study generalized right (principally) quasi Baer skew monoid rings. Examples to illustrate and delimit the results are provided.

Keywords: APP rings; generalized quasi Baer rings; generalized principally quasi Baer rings; primary; skew monoid rings

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1. Introduction

Let F' be a free monoid with a generating set $X = \{u_1, \ldots, u_t\}$ and identity element e; and $F = F' \cup \{0\}$. Let M be a factor of F which setting certain monomials in X to 0. Suppose M is nilpotent of the nth order. That's mean $\alpha^n = 0$ for all $\alpha \in M$ and there exist some $\beta \in M$ and $\beta^{n-1} \neq 0$. Let R be a ring with 1 an σ be an automorphism of R. The skew monoid ring $R[M;\sigma]$ was studied for the first time in [10]. Clearly, the addition operation of this ring is as usual and the multiplication operation being skewed by the rule $u_i r = \sigma(r)u_i$ for all $i = 1, \ldots, t$. So elements of $R[M;\sigma]$ are as follows:

$$ae + \sum_{1\leq k\leq t} a_k u_k + \sum_{1\leq k_1,k_2\leq t} a_{k_1k_2}u_{k_1}u_{k_2} + \cdots + \sum_{1\leq k_1,\ldots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}}.$$

[‡]Corresponding author.

In [13], the authors introduced generalized (principally) quasi Baer rings. A ring R is called *generalized right quasi Baer* if satisfies the following condition:

$$\forall\, I \leq_r R, \quad \exists\, n \in \mathbb{N} \quad \text{and} \quad e = e^2 \in R \quad \text{such that } r_R(I^n) = eR.$$

A ring R is called *generalized right principally quasi Baer* whenever the above condition holds for all principal right ideals of R. Given a fixed positive integer n, a ring R is called *n-generalized right* (*principally*) *quasi Baer* if the natural number n, for all ideals satisfies the above condition. Left cases may be defined analogously. By [13, Proposition 2.2(i)] we have the following:

(1) A semiprime ring R is n-generalized right quasi Baer if and only if R is quasi Baer. (i.e. the right (equiv. left) annihilator of every right (equiv. left) ideal of R is generated as a right (equiv. left) ideal by an idempotent. See [6].)
(2) A semiprime ring R is n-generalized right principally quasi Baer if and only if R is right principally quasi Baer. (i.e. the right annihilator of every principal right ideal is generated by an idempotent. See [2].)

In this paper, by applying a rigid constraint on endomorphism σ. We determine semicentral idempotents of $R[M;\sigma]$ in terms of the semicentral idempotents of the base ring R and with the help of this result, we express the results with the generalized quasi Baer and generalized principally quasi Baer property of these rings. Also, we have shown that for σ-weakly rigid ring R, two properties of being left primary and semicentral reduced are equivalent for both rings R and $R[M;\sigma]$. It should be noted in this paper, $r_R(X)$ denotes the right annihilator of the subset X in a ring R and for an element α in the skew monoid ring $R[M;\sigma]$, $\mathfrak{C}_\alpha$ denotes the set of coefficients of α.

2. Results

In general, in order to better study any algebraic structure, including the skew monoid ring $R[M;\sigma]$, in addition to applying conditions on the base ring R, we need to apply some restrictions on σ. One of the common restrictions is

$$a\sigma(Rb) = 0 \Leftrightarrow aRb = 0, \quad \forall\, a, b \in R.$$

A ring R with the above restriction is called *σ-weakly rigid*, by [14]. Clearly, prime rings with any automorphism σ are examples of σ-weakly rigid rings. In [4], the set of right and left semicentral idempotents of R were symbolized by $\mathcal{S}_r(R)$ and $\mathcal{S}_\ell(R)$, respectively. In fact,

$$\mathcal{S}_r(R) = \{e \in R : e^2 = e \text{ and } ex = exe, \forall\, x \in R\},$$

$$\mathcal{S}_\ell(R) = \{e \in R : e^2 = e \text{ and } xe = exe, \forall\, x \in R\}.$$

Clearly, for semiprime or abelian ring R, we have $\mathcal{S}_r(R) = \mathcal{S}_\ell(R) = \mathbf{B}(R)$, where $\mathbf{B}(R)$ is the set of central idempotents of R.

As a first result, we characterize the semicentral idempotents which are central in $R[M;\sigma]$ in terms of the semicentral idempotents of R.

Proposition 2.1. *Let R be a σ-weakly rigid ring. Then*:

(i) $\mathcal{S}_r(R) = \boldsymbol{B}(R)$ *if and only if* $\mathcal{S}_r(R[M;\sigma]) = \boldsymbol{B}(R[M;\sigma])$.
(ii) $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$ *if and only if* $\mathcal{S}_\ell(R[M;\sigma]) = \boldsymbol{B}(R[M;\sigma])$.

Moreover, If the subsets of right (respectively, left) semicentral idempotents and central idempotents of R are the same, then every central idempotent in the ring $R[M;\sigma]$ is of the form fe, where $f^2 = f \in \boldsymbol{B}(R)$.

Proof. We only prove (i) and the proof of the other case is similar. First assume $\mathcal{S}_r(R) = \mathbf{B}(R)$ and

$$\mathfrak{F} = fe + \sum_{1\le k\le t} f_k u_k + \sum_{1\le k_1,k_2\le t} f_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\dots,k_{n-1}\le t} f_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}$$

in $\mathcal{S}_r(R[M;\sigma])$. Thus, $\mathfrak{F}R[M;\sigma](e-\mathfrak{F}) = 0$ and consequently $\mathfrak{F}R(e-\mathfrak{F}) = 0$. Therefore, we obtain the following equations:

(1) $fR(1-f) = 0$;
(2) $fRf_k = f_k R\sigma(1-f) \qquad 1 \le k \le t$;
(3) $fRf_{k_1k_2} = f_{k_1k_2} R\sigma^2(1-f) \qquad 1 \le k_1, k_2 \le t$;
$\vdots$

Considering Eq. (1), we obtain $f \in \mathcal{S}_r(R) = \mathbf{B}(R)$. Now, by multiplying Eq. (2) by f from the left-hand side, we have $fRf_k = ff_kR\sigma(1-f) = 0$. This equation is obtained from Eq. (1) and σ-weakly rigidity of R. Hence, $f_kR(1-f) = 0$, by Eq. (2) and σ-weakly rigidity of R and so

$$f_k = f_k f = f f_k \in fRf_k = 0.$$

Continuing this method, we see that $\mathfrak{F} = fe$. On the other hand, $fR(1-f) = (1-f)Rf = 0$ implies that $f(1-\sigma(f)) = (1-f)\sigma(f) = 0$ and consequently $f = \sigma(f)$. By the above argument, one can easily check that $\mathfrak{F} \in \mathbf{B}(R[M;\sigma])$ and so $\mathcal{S}_r(R[M;\sigma]) = \mathbf{B}(R[M;\sigma])$. Conversely, let $\mathcal{S}_r(R[M;\sigma]) = \mathbf{B}(R[M;\sigma])$ and $f \in \mathcal{S}_r(R)$. Hence, $fR(1-f) = 0$. Then the σ-weakly rigidity of R implies that $(fe)R[M;\sigma]((1-f)e) = 0$. Thus, $fe \in \mathcal{S}_r(R[M;\sigma]) = \mathbf{B}(R[M;\sigma])$. Therefore, $f \in \mathbf{B}(R)$ and the result follows. □

Note that f is left semicentral if and only if $1-f$ is right semicentral. So $\mathcal{S}_r(R) = \{0,1\}$ is equivalent to $\mathcal{S}_\ell(R) = \{0,1\}$. A ring R is said to be *semicentral reduced* if $\mathcal{S}_\ell(R) = \{0,1\}$. As an immediate consequence of Proposition 2.1, we obtain the following.

Corollary 2.2. *Let R be a σ-weakly rigid ring. Then R is semicentral reduced if and only if so is $R[M;\sigma]$.*

In the following, we state a sufficient condition for the skew monoid ring $R[M;\sigma]$ to be n-generalized right (principally) quasi Baer. For this purpose, we need the following lemma about the semiprime or right APP rings. According to Liu and Zhao [12], a ring R is called *right APP* if $r_R(aR)$ is left s-unital (i.e. for each $a \in r_R(aR)$ there is an $x \in r_R(aR)$ such that $xa = a$).

Lemma 2.3. *Let R be a semiprime or right APP σ-weakly rigid ring and α_i be elements of $T = R[M;\sigma]$ for each $1 \leq i \leq n$. If $\alpha_1 T \alpha_2 T \cdots T \alpha_n = 0$, then $p_1 R p_2 \cdots R p_n = 0$ for each $p_i \in \mathfrak{C}_{\alpha_i}$ and $1 \leq i \leq n$.*

Proof. The proof can be easily done by induction and considering that the lemma is proved in [11, Proposition 3.1] in the case of $n = 2$. □

Theorem 2.4. *Let R be a (right principally) quasi Baer σ-weakly rigid ring with $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$. Then $R[M;\sigma]$ is n-generalized right (principally) quasi Baer.*

Proof. Suppose that I is a right ideal of $T = R[M;\sigma]$. Let J be the right ideal of R consists of coefficients of identity e of all elements of I. Since R is quasi Baer, there exists a left semicentral idempotent $f \in \mathcal{S}_\ell(R)$ such that $r_R(J) = fR$. We claim that $r_T(I^n) = (fe)T$. With a simple calculation and considering σ-weakly rigidity of R and $\sigma(f) = f \in \mathbf{B}(R)$, we have $(fe)T \subseteq r_T(I^n)$. Now, assume

$$\beta = be + \sum_{1 \leq k \leq t} b_k u_k + \sum_{1 \leq k_1, k_2 \leq t} b_{k_1 k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1 \leq k_1, \ldots, k_{n-1} \leq t} b_{k_1 \cdots k_{n-1}} u_{k_1} \cdots u_{k_{n-1}}$$

is an arbitrary element of $r_T(I^n)$. Then, for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ we obtain $\alpha_1 T \alpha_2 T \cdots T \alpha_n T \beta = 0$. Since R is right APP, Lemma 2.3 implies that $a_1 R a_2 R \cdots R a_n R \mathfrak{b} = 0$ for each $a_i \in J$ and $\mathfrak{b} \in \mathfrak{C}_\beta$. Therefore, $\mathfrak{b} \in r_R(J^n)$ and hence $J^n \mathfrak{b} = J(J^{n-1}\mathfrak{b}) = 0$. So $J^{n-1}\mathfrak{b} = J^{n-1}\mathfrak{b} f = J^{n-1} f \mathfrak{b} = 0$. By repeating this technique, we get $\mathfrak{b} \in r_R(J)$ and thus $\mathfrak{b} = f\mathfrak{b}$. It follows that $r_T(I^n) \subseteq (fe)T$. Therefore, the ring $R[M;\sigma]$ is n-generalized right quasi Baer. □

In [5], the authors introduced a *skew triangular matrix ring* $T_n(R,\sigma)$ as a set of all triangular matrices with usual addition and multiplication which is skewed by the rule $E_{ij} r = \sigma^{j-i}(r) E_{ij}$, where E_{ij} is the elementary matrix. Actually, for each (a_{ij}) and (b_{ij}) in $T_n(R,\sigma)$, we have

$$(a_{ij})(b_{ij}) = (c_{ij}) \quad c_{ij} = a_{ii} b_{ij} + a_{i,i+1} \sigma(b_{i+1,j}) + \cdots + a_{ij} \sigma^{j-i}(b_{jj})$$

for each $i \leq j$. Just note that to define the well-defined multiplication, we need $\sigma(1) = 1$. Three following important subrings of $T_n(R,\sigma)$ that are widely found in the literature are examples of $R[M;\sigma]$. (For more details see [10, 15]).

$$A(R,n,\sigma) = \{A : |D_i| = 1, \forall\, i = 1, \ldots, [n/2]\};$$

$$B(R,n,\sigma) = \{A + rE_{1,n/2} : A \in A(R,n,\sigma), r \in R\}; \quad (n \text{ is even})$$

$$T(R,n,\sigma) = \{A : |D_i| = 1, \forall\, i = 1,\dots,n\};$$

where

$$D_i = \{a_{1,i}, a_{2,i+1}, \dots, a_{n-i+1,n}\}, \quad \forall\, i = 1,\dots,n.$$

Note that in case $n = 2$ and $\sigma = id_R$, the ring $T(R,n,\sigma)$ is the trivial extension $T(R,R)$. So we have the following corollary. It should be noted that the first item of this result was stated in [13, Theorem 3.2].

Corollary 2.5. *Let R be a (right principally) quasi Baer ring ring with $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$. Then we have the following statements:*

(a) *The ring $S(R,n,id_R)$ is n-generalized right (principally) quasi Baer.*
(b) *The ring $A(R,n,id_R)$ is n-generalized right (principally) quasi Baer.*
(c) *The ring $B(R,n,id_R)$ is n-generalized right (principally) quasi Baer.*
(d) *The ring $T(R,n,id_R)$ is n-generalized right (principally) quasi Baer.*
(e) *The ring $R[x]/(x^n)$ is n-generalized right (principally) quasi Baer.*
(f) *The trivial extension $T(R,R)$ is n-generalized right (principally) quasi Baer.*

The next example allows us to construct numerous examples of generalized right (principally) quasi Baer rings which are not semiprime.

Example 2.6. Let R be a prime ring. From [1, Lemma 4.2], R is quasi Baer and semicentral reduced. Suppose σ is an automorphism of R. Clearly, R is σ-weakly rigid. Therefore, Theorem 2.4 implies that $R[M;\sigma]$ is n-generalized right (principally) quasi Baer.

In the following example, we see that for a quasi Baer ring R, $R[M;\sigma]$ is not necessarily (right principally) quasi Baer, in general.

Example 2.7. Suppose that R is a prime ring with an automorphism σ. Let M be a free monoid generated by $\{u,v\}$ with 0 added and the relation

$$u^2 = v^2 = uv = vu = 0$$

and $T = R[M;\sigma]$. Then $r_T(uT) = Ru + Rv$ is not generated by an idempotent of $R[M;\sigma]$. This is because that T is reduced semicentral by Corollary 2.2. Therefore, $R[M;\sigma]$ is not (right principally) quasi Baer.

Now, we state the sufficient condition that the generalized right (principally) quasi Baerness of the base ring R and the skew monoid ring $R[M;\sigma]$ is equivalent.

Theorem 2.8. *Let R be a σ-weakly rigid ring with $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$. Then R is generalized right (principally) quasi Baer if and only if so is $R[M;\sigma]$.*

Proof. Assume that R is generalized right quasi Baer and I is a right ideal of $T = R[M;\sigma]$. Let J be the right ideal of R consists of coefficients of identity e

of all elements of I. So, there exist positive integer m and $f \in \mathcal{S}_\ell(R)$ such that $r_R(J^m) = fR$. We have

$$r_R(J^m) = r_R(J^{m+1}) = r_R(J^{m+2}) = \cdots.$$

This is because that $J^{m+1}x = 0$ implies that $Jx \in r_R(J^m) = fR$ and consequently $Jx = fJx = Jfx$, since f is central. Thus, $J^m x = J^m(fx) = 0$. We will show that $r_T(I^{nm}) = (fe)T$. Clearly, by considering σ-weakly rigidity of R and $f = \sigma(f) \in \mathbf{B}(R)$ we get $(I^{nm})(fe) = 0$ and this implies that $(fe)T \subseteq r_T(I^{nm})$. Next, let

$$\beta = be + \sum_{1\le k\le t} b_k u_k + \sum_{1\le k_1,k_2\le t} b_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} b_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}$$

in $r_T(I^{nm})$. Thus, $\alpha T\beta = 0$ for each

$$\alpha = ae + \sum_{1\le k\le t} a_k u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}$$

in I^{nm}.

So for each $r \in R$ we have $\alpha(re)\beta = 0$ and thus we obtain the following equations:

(1) $arb = 0$;
(2) $arb_k + a_k\sigma(r)\sigma(b) = 0$ $\quad 1 \le k \le t$;
(3) $arb_{k_1k_2} + a_{k_1}\sigma(r)\sigma(b_{k_2}) + a_{k_1k_2}\sigma^2(r)\sigma^2(b) = 0$ $\quad 1 \le k_1, k_2 \le t$;
$\vdots$

Considering Eq. (1), we obtain $aRb = 0$. Therefore, $b \in r_R(J^{mn}) = fR$ and so $b = fb$. Now by replacing r with rar_1 for any $r_1 \in R$ in Eq. (2), we get $arar_1b_k = 0$. Thus, $b_k \in r_R(J^{2mn}) = fR$ and so $b_k = fb_k$ for all $1 \le k \le t$. Next by replacing r with rar_1ar_2 for any r_1 and r_2 in R in Eq. (3), we have $arar_1ar_2b_{k_1k_2} = 0$. Hence, $b_{k_1k_2} \in r_R(J^{3mn}) = fR$ and so $b_{k_1k_2} = fb_{k_1k_2}$ for all $1 \le k_1, k_2 \le t$. By continuing in this way, one can see that $\mathfrak{b} = f\mathfrak{b}$ for all $\mathfrak{b} \in \mathfrak{C}_\beta$. Thus, $\beta = (fe)\beta \in (fe)T$ and this proves that T is a generalized right quasi Baer ring. Conversely, assume T is a generalized right quasi Baer ring. Let I be a right ideal of R and $I[M;\sigma] = \{\mathfrak{I} \in T \mid \mathfrak{C}_\mathfrak{I} \subseteq I\}$. Clearly, $I[M;\sigma]$ is a right ideal of T. So there exist positive integer m and $\alpha^2 = \alpha \in T$ such that $r_T((I[M;\sigma])^m) = \alpha T$. Let

$$\alpha = ae + \sum_{1\le k\le t} a_k u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}.$$

Clearly, a is an idempotent element of R. Since $(I[M;\sigma])^m\alpha = 0$, we have $I^m a = 0$ and hence $a \in r_R(I^m)$. Thus, $aR \subseteq r_R(I^m)$. Next, let $r \in r_R(I^m)$. Then $re \in r_T((I[M;\sigma])^m)$, by σ-weakly rigidity of R. Thus, $re = \alpha(re)$ and consequently $r = ar \in aR$. Therefore, $r_R(I^m) = aR$. So R is a generalized right quasi Baer ring and the proof is complete. □

Note that weakly rigidity of R and Proposition 2.1 cause the left-hand version of Theorems 2.4 and 2.8 to be maintained

Now, by using the quasi Baer ring R and using two monoids M and N, we will obtain a wider range of generalized right (principally) quasi Baer rings. Notice that for any endomorphism μ with $\mu\sigma = \sigma\mu$ and $\mu(1_R) = 1_R$, we can define the endomorphism $\bar{\mu} : R[M;\sigma] \to R[M;\sigma]$ by $\mu(\sum_{g\in M} r_g g) = \sum_{g\in M} \mu(r_g)g$.

Corollary 2.9. *Let R be an arbitrary ring with two weakly rigid automorphisms σ and μ with $\sigma\mu = \mu\sigma$. If R is quasi Baer and $\mathcal{S}_\ell(R) = \boldsymbol{B}(R)$, then $R[M;\sigma][N;\bar{\mu}]$ is generalized right quasi Baer.*

Proof. Note that quasi Baer rings are APP. So $\alpha R[M;\sigma]\beta = 0$ if and only if $aRb = 0$ if and only if $aR\mu(b) = 0$ if and only if $\alpha R[M;\sigma]\bar{\mu}(\beta) = 0$, by Lemma 2.3. This implies that $R[M;\sigma]$ is $\bar{\mu}$-weakly rigid. Hence, the proof is clear, by Proposition 2.1 and Theorem 2.8. □

Recall from [7–9], a ring R is *left primary* if whenever A and B are ideals of R with $AB = 0$, then $B = 0$ or $A^n = 0$ for some positive integer n. Primary rings are a generalization of prime rings and nilpotent rings. By [3, Proposition 3.6(i)], a ring R is left primary if and only if R is semicentral reduced and generalized right quasi Baer. So with the help of Theorem 2.8 and Corollary 2.2, we can produce a wide range of left primary rings.

Theorem 2.10. *Let R be a σ-weakly rigid ring, where σ is an automorphism of R. Then R is left primary if and only if so is $R[M;\sigma]$.*

We finish this paper by the following example that allows us to construct enormous examples of generalized right quasi Baer rings which are not left primary.

Example 2.11. Let R be a non-prime σ-weakly rigid quasi Baer ring with $\mathcal{S}_\ell(R) = \mathbf{B}(R)$. The ring $R[M;\sigma]$ is generalized right quasi Baer, by Theorem 2.8 but it is not left primary, since R is not semicentral reduced.

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