Hom-Lie algebroids associated to Hom-Poisson manifolds

Hassan Arianpoor

Tafresh University, Department of Mathematics

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Introduction

- In Hom algebras, the identities defining the structures are twisted by homomorphisms. Such algebras appeared in some q-deformations of the Witt and the Virasoro algebras.
- Motivated by their generalization, Hartwing, Larsson and Silvestrov introduced the notion of Hom-Lie algebras as part of the study of σ-dreivations of an associated algebra. In the class of Hom-Lie algebras, the Jacobi identity is twisted by a homomorphism, called the Hom-Jocobi identity.
- Makhlouf and Silvestrov have modified the definition of a Hom-Lie algebra when we start with a nonassociative algebra and an automorphism of it.
- Hom-Lie algebras are widely studied in representation and cohomology theory, deformation theory and bialgebra theory.

Introduction

- The notion of a Lie algebroid was introduced by Pradines as a generalization of Lie algebras and tangent bundles.
- Gengoux and Teles introduced the notion of a Hom-Lie algebroid with the help of a Hom-Gerstenhaber algebra.
- There is a one to one correspondence between Hom-Gerstenhaber algebra structures on $\Gamma(\wedge^\bullet \mathcal{A})$ and Hom-Lie algebroid stuctures on a vector bundle $\mathcal{A}$.
- We introduce the notion of a Hom-bundle by any diffeomorphism φ on M to make the definition of a Hom-Lie algebroid on the pull back $\varphi^! TM$ of the Lie algebroid TM.
- By introducing the notion of a Hom-Poisson tensor which is equivalent to a purely Hom-Poisson algebra structure on $C^\infty(M)$, we show that there is a Hom-Lie algebroid structure on $\varphi^! TM$, associated to a Hom-Poisson manifold.

Overview of paper topic

In this paper, we modify the definition of Hom-Lie algebroids introduced by Gengoux and Teles, by using the concept of Hom bundles. Then we give the notion of Hom-Poisson manifolds and show that there is a Hom-Lie algebroid structure on the pullback of the cotangent bundle of a Hom-Poisson manifold.

Let $\mathfrak{g}$ be a vector space and $[.,.] : \wedge^2\mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, a linear map. An automorphism of $(\mathfrak{g}, [.,.])$ is a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

Definition

A Hom-Lie algebra is a triple $(\mathfrak{g}, [.,.], \alpha)$ with a vector space $\mathfrak{g}$ equipped with a skew symmetric bilinear map $[.,.] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ and an automorphism α of $(\mathfrak{g}, [.,.])$ satisfying the Hom-Jacobi identity: $\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0, \ \forall x, y, z \in \mathfrak{g}$.

Given a vector space (Lie algebra) $\mathfrak{g}$equipped with a skew symmetric bilinear map $[.,.] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ and an automorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ of $(\mathfrak{g}, [.,.])$, defined $[.,.]_\alpha : \wedge^2\mathfrak{g} \to \mathfrak{g}$ by $[x, y]_\alpha = \alpha([x, y]), \ \forall x, y \in \mathfrak{g}$. Then $(\mathfrak{g}, [.,.]_\alpha, \alpha)$ is a Hom-Lie algebra iff the restriction of $[.,.]$ to the image of α^2 is a Lie bracket.

Definition

A Hom-associative algebra is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a vector space $\mathcal{A}$, a bilinear map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and an automorphism α of $(\mathcal{A}, \mu)$ satisfying the Hom-associativity: $\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)), \ \forall x, y, z \in \mathcal{A}$.

- Given $(\mathcal{A}, \mu)$ an associative algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ an algebra automorphism, $(\mathcal{A}, \alpha \circ \mu, \alpha)$ is a Hom-associative algebra.
- A Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$ generates a Hom-Lie algebra $(\mathcal{A}, [., .], \alpha)$, when $[x, y] = \mu(x, y) - \mu(y, x)$.

A homomorphism of Hom-associative algebras (resp. Hom-Lie algebras) $(\mathfrak{g}, \mu_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{h}, \mu_{\mathfrak{h}}, \beta)$ (resp. $(\mathfrak{g}, [., .]_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{h}, [., .]_{\mathfrak{h}}, \beta)$) is a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ such that $\psi(\mu_{\mathfrak{g}}(x, y) = \mu_{\mathfrak{h}}(\psi(x), \psi(y)))$ (resp.$\psi([x, y]_{\mathfrak{g}}) = [\psi(x), \psi(y)]_{\mathfrak{h}}$) and $\psi(\alpha(x)) = \beta(\psi(x)), \ \forall x, y \in \mathfrak{g}$.

Example

Let $\mathcal{A}$ be a 2-vector space with a basis $\{e_1, e_2\}$. A multiplication μ and a linear map α on $\mathcal{A}$ are given by $\mu(e_1, e_1) = e_1$, $\mu(e_i, e_j) = e_2$, if $(i, j) \neq (1, 1)$ and $\alpha(e_1) = \lambda e_1 + \gamma e_2,\ \alpha(e_2) = (\lambda + \gamma)e_2$, where $\lambda, \gamma \neq 0$. Then $(\mathcal{A}, \mu, \alpha)$ is a Hom-associative algebra and $(\mathcal{A}, [.,.], \alpha)$ is a Hom-Lie algebra with $[e_i, e_j] = \mu(e_i, e_j) - \mu(e_j, e_i)$.

Definition

A Hom-Poisson algebra is a quadruple $(\mathcal{A}, \mu, \{.,.\}, \alpha)$ with a vector space $\mathcal{A}$, bilinear maps $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\{.,.\} : \wedge^2\mathcal{A} \to \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \to \mathcal{A}$ such that satisfies:

- $(\mathcal{A}, \mu, \alpha)$ is a commutative Hom-associative algebra,
- $(\mathcal{A}, \{.,.\}, \alpha)$ is a Hom-Lie algebra,
- $\{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\{x, y\}, \alpha(z))$, for all $x, y, z \in \mathcal{A}$.

Definition

A purely Hom-Poisson algebra is a quadruple $(\mathcal{A}, \mu, \{.,.\}, \alpha)$ consisting of a vector space $\mathcal{A}$, bilinear maps $\mu : \wedge^2\mathcal{A} \to \mathcal{A}$ and $\{.,.\} : \wedge^2\mathcal{A} \to \mathcal{A}$ and a linear map $\alpha : \mathcal{A} \to \mathcal{A}$ such that satisfies:

- $(\mathcal{A}, \mu)$ is a commutative associative algebra,
- $(\mathcal{A}, \{.,.\}, \alpha)$ is a Hom-Lie algebra,
- $\{x, \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\{x, y\}, \alpha(z))$, for all $x, y, z \in \mathcal{A}$.

Let $(\mathcal{A}, \mu, \{.,.\})$ be a Poisson algebra and $\alpha : \mathcal{A} \to \mathcal{A}$ a Poisson automorphism. $(\mathcal{A}, \mu_\alpha = \alpha \circ \mu, \{.,.\}_\alpha = \alpha \circ \{.,.\}, \alpha)$ is a Hom-Poisson algebra iff $\{.,.\}$ (resp. μ) is a Lie bracket (resp. an associative product) when restricted to the image of α^2 and $(\mathcal{A}, \mu, \{.,.\}_\alpha = \alpha \circ \{.,.\}, \alpha)$ is a purely Hom-Poisson algebra.

Example

- Let $\mathcal{A}$ be a 3-vector space with a basis $\{x_1, x_2, x_3\}$. The following multiplication μ, bracket $\{.,.\}$ and a linear map α on $\mathcal{A}$ define a Hom-Poisson algebra: $\mu(x_1, x_1) = x_1,\ \mu(x_1, x_2) = \mu(x_2, x_1) = x_3$. For $a, b, c, d \neq 0$ let $\{x_1, x_2\} = ax_2 + bx_3,\ \{x_1, x_3\} = cx_2 + dx_3$ and $\alpha(x_1) = \lambda_1 x_2 + \lambda_2 x_3,\ \alpha(x_2) = \lambda_3 x_2 + \lambda_4 x_3,\ \alpha(x_3) = \lambda_5 x_2 + \lambda_6 x_3$.
- Given (M, π) a Poisson manifold with Poisson bivector field π and $\varphi : M \to M$ a smooth map, then there is a Hom-Poisson structure on $C^\infty(M)$ if φ preserves the bivector field π: $\pi_{\varphi(m)} = (\wedge^2 T_m\varphi)(\pi_m),\ \forall m \in M$ and that the Schouten-Nijenhuis bracket $[\pi, \pi]$ vanishes on $\varphi^2(M) \subset M$. Hence, $(C^\infty(M), \mu_{\varphi^*} = \varphi^* \circ \mu, \{.,.\}_{\varphi^*} = \varphi^* \circ \{.,.\}, \varphi^*)$ is a Hom-Poisson algebra with the usual product μ on $C^\infty(M)$ and $(C^\infty(M), \mu, \{.,.\}_{\varphi^*} = \varphi^* \circ \{.,.\}, \varphi^*)$ is a purely Hom-Poisson algebra. (M, π, φ) is called a Hom-Poisson manifold.

Definition

A pair $(\mathcal{A} = \oplus_{i\in\mathbb{N}}\mathcal{A}_i, [.,.])$ is a $\mathbb{N}$-graded Lie algebra if

- $\mathcal{A}$ is a graded algebra in which $[\mathcal{A}^n, \mathcal{A}^m] \subset \mathcal{A}^{n+m}$,
- the bracket $[.,.]$ in $\mathcal{A}$ is graded skew summetric: $[x, y] = (-1)^{pq}[y, x]$, for all $x \in \mathcal{A}^p,\ y \in \mathcal{A}^q$,
- it satisfies the graded Jacobi identity: $\circlearrowleft_{x,y,z} (-1)^{pq}[x, [y, z]] = 0$, for all $x \in \mathcal{A}^p,\ y \in \mathcal{A}^r,\ z \in \mathcal{A}^q$.

Definition

A Hom-Gerstenhaber algebra is a graded Hom-Lie algebra $(\mathfrak{g}, \wedge, [[.,.]], \alpha)$ where $(\mathfrak{g} = \oplus_{i\in\mathbb{N}}\mathfrak{g}_i, \wedge)$ is a graded associative algebra, α is an automorphism of $(\mathfrak{g}, \wedge)$ of degree 0 and $[[.,.]] : \wedge^2\mathfrak{g} \to \mathfrak{g}$ is a linear map of degree -1 satisfying the following Hom-Leibniz rule for all $X \in \mathfrak{g}_i,\ Y \in \mathfrak{g}_j,\ Z \in \mathfrak{g}$:

$$[[X, Y \wedge Z]] = [[X, Y]] \wedge \alpha(Z) + (-1)^{(i-1)j}\alpha(Y) \wedge [[X, Z]]. \quad (1)$$

Definition

A representation (V, β, ρ) of a Hom-Lie algebra $(\mathfrak{g}, [.,.], \alpha)$ on a vector space V with respect to $\beta \in \mathfrak{gl}(V)$ is a linear map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ such that for all $x, y \in \mathfrak{g}$ the following satisfy:

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x),$$

$$\rho([x, y]) \circ \beta = \rho(\alpha(x) \circ \rho(y) - \rho(\alpha(y) \circ \rho(x). \tag{2}$$

Example

Let $(\mathfrak{g}, [.,.], \alpha)$ be a Hom-Lie algebra. The linear map α can be extended to a linear map $\alpha : \wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g}$. The schouten bracket $[[.,.]] : \wedge^\bullet \mathfrak{g} \otimes \wedge^\bullet \mathfrak{g} \to \wedge^\bullet \mathfrak{g}$ is obtained from $[.,.]$ for all $x_i, y_j \in \mathfrak{g}$ by

$$[[x_1 \wedge \ldots \wedge x_m, y_1 \wedge \ldots \wedge y_n]] =$$

$$\sum_{i,j} (-1)^{i+j} [x_i, y_j] \wedge \alpha(x_1 \wedge \ldots \hat{x}_i \ldots \wedge x_m \wedge y_1 \wedge \ldots \hat{y}_j \ldots \wedge y_n),$$

$(\wedge^\bullet \mathfrak{g}, \wedge, [[.,.]], \alpha)$ is a Hom-Gerstenhaber algebra. The adjoint representation of $\mathfrak{g}$ on $\wedge^\bullet \mathfrak{g}$ is given by $ad_x Y = [[x, Y]]$ for all $x \in \mathfrak{g}$, $Y \in \wedge^k \mathfrak{g}$.

Given an algebra automorphism σ of a commutative algebra $\mathcal{A}$, a σ-derivation on $\mathcal{A}$ is a linear map $D(ab) = D(a)\sigma(b) + \sigma(a)D(b)$, for all $a, b \in \mathcal{A}$. We denote by $Der_\sigma(\mathcal{A})$ the set of all σ-derivations.

Theorem

Let $(\mathcal{A} = \oplus_{i\in\mathbb{N}}\mathcal{A}_i, \wedge, [[.,.]], \alpha)$ be a Hom-Grestenhaber algebra. Then $(\mathcal{A}_1, \wedge, [[.,.]]|_{\mathcal{A}_1\times\mathcal{A}_1}, \alpha|_{\mathcal{A}_1})$ is a Hom-Lie algebra and $(\mathcal{A}_0, \rho, \alpha|_{\mathcal{A}_0})$ is its representaion in which for $f \in \mathcal{A}_0$ and $X \in \mathcal{A}_1$, the map $\rho : \mathcal{A}_1 \to End(\mathcal{A}_0)$ is defined by $\rho(X)(f) = [[X, f]]$ and a $\alpha|_{\mathcal{A}_0}$-derivation of $\mathcal{A}_0$ is given as $f \mapsto \rho(X)(f)$.

Definition

A Hom-bundle $(\mathcal{A}, \varphi, \alpha)$ is a vector bundle $\mathcal{A}$ over M, equipped with a smooth map φ on M and a linear map $\alpha : \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ such that $\alpha(fX) = \varphi^*(f)\alpha(X), \ \forall X \in \Gamma(\mathcal{A}), \ f \in C^\infty(M)$.

For the pullback bundle $\varphi^!\mathcal{A}$ of $\mathcal{A}$ along φ, the corresponding pullback section $y^!$ at a point $m \in M$ is given by $y^!(m) = y(\varphi(m))$ where $y \in \Gamma(\mathcal{A})$.

Example

For any bundle map $\Phi : \varphi^!\mathcal{A} \to \mathcal{A}$, defined $\phi_{\varphi^!\mathcal{A}} : \Gamma(\varphi^!\mathcal{A}) \to \Gamma(\varphi^!\mathcal{A})$ by $\phi_{\varphi^!\mathcal{A}}(x) = (\Phi(x))^!$ for all $x \in \Gamma(\varphi^!\mathcal{A})$. Then, $(\varphi^!\mathcal{A}, \varphi, \phi_{\varphi^!\mathcal{A}})$ is a Hom-bundle.

Example

Let φ be a smooth map on M. The pullback map φ^* is a homomorphism on $C^\infty(M)$ as $\varphi^*(fg) = \varphi^*(f)\varphi^*(g)$. $\Gamma(\varphi^! TM)$ can be identified with $Der_\sigma(C^\infty(M))$, by mapping $X \in \Gamma(\varphi^! TM)$ to the φ^*-derivation which maps f to $X_m d_{\varphi(m)} f$ at $m \in M$ such that $X(fg) = X(f)\varphi^*(g) + \varphi^*(f)X(g)$. The map Ad_{φ^*} on $\Gamma(\varphi^! TM)$ is defined by $Ad_{\varphi^*}(X) = \varphi^* \circ X \circ (\varphi^*)^{-1}$, for all $X \in \Gamma(\varphi^! TM)$. Then, $(\varphi^! TM, \varphi, Ad_{\varphi^*})$ is a Hom-bundle.

Definition

A Hom-Lie algebroid structure on a Hom-bundle $(\mathcal{A}, \varphi, \alpha)$ is a quintuple $(\mathcal{A}, \varphi, [.,.], \rho, \alpha)$ consisting of a Hom-Lie algebra $(\Gamma(\mathcal{A}), [.,.], \alpha)$ and the anchor map $\rho : \varphi^! \mathcal{A} \to \varphi^!(TM)$ such that for all $X, Y \in \Gamma(\mathcal{A})$, $f \in C^\infty(M)$ satisfy the following:
(i) $\alpha(fX) = \varphi^*(f)\alpha(X)$, (ii) $(C^\infty(M), \rho, \varphi^*)$ is a representation of $(\Gamma(\mathcal{A}), [.,.], \alpha)$, (iii) the Hom-Leibniz identity holds:
$[X, fY] = \varphi^*(f)[X, Y] + \rho(X)(f)\alpha(Y)$.

- Linear endomorphisms α are in one to one correspondence with vector bundle morphisms from $\varphi^!\mathcal{A}$ to $\mathcal{A}$ over the identity of M. A section of the pullback bundle $\varphi^!\mathcal{A}$ is given by mapping $m \in M$ to $X_{\varphi(m)} \in \mathcal{A}_{\varphi(m)} \simeq (\varphi^!\mathcal{A})_m$ for $X \in \Gamma(\mathcal{A})$. Applying a vector bundle morphism from $\varphi^!\mathcal{A}$ to $\mathcal{A}$ over the identity of M to X, yields a section of $\mathcal{A}$ and α satisfies (i).
- The Hom-Leibnizidentity implies that for sections X, Y of $\mathcal{A}$, the value of $[X, Y]$ at a point $m \in M$ depends only on the corresponding one on $\varphi^!\mathcal{A}$ at $\varphi(m)$.
- The value of $\rho(X)(f)$ at $m \in M$ is equal to $\langle d_{\varphi(m)}f, \rho_m(X_{\varphi(m)})\rangle$, where $X_{\varphi(m)}$ is the value of $X \in \Gamma(\mathcal{A})$ at $\varphi(m) \in M$ and $\rho_m : (\varphi^!\mathcal{A})_m \simeq \mathcal{A}_{\varphi(m)} \to (\varphi^! TM)_m \simeq T_{\varphi(m)}M$ is the anchor map evaluated at $m \in M$.

Theorem

Let $\varphi : M \to M$ be a diffeomorphism. Then, $(\varphi^! TM, \varphi, Ad_{\varphi^*}, [.,.]_{\varphi^*}, id)$ is a Hom-Lie algebroid where Ad_{φ^*} is given in the before example and a skew symmetric bileaner operation $[.,.]_{\varphi^*} : \wedge^2\Gamma(\varphi^! TM) \to \Gamma(\varphi^! TM)$, defined by $[X, Y]_{\varphi^*} = \varphi^* \circ X \circ (\varphi^*)^{-1} \circ Y \circ (\varphi^*)^{-1} - \varphi^* \circ Y \circ (\varphi^*)^{-1} \circ X \circ (\varphi^*)^{-1}$.

Let $(C^\infty(M), \{.,.\}, \varphi^*)$ be a purely Hom-Poisson algebra on $C^\infty(M)$. The Hom-Leibniz rule implies that $\{f,.\}$ is a φ^*-derivation on $C^\infty(M)$. There exists a bivector field (bisection) $\pi \in \Gamma(\wedge^2\varphi^! TM)$ such that $\{f, g\} = \pi(df, dg)$ for all $f, g \in C^\infty(M)$. The bisection π on M is called a Hom-Poisson tensor if $[[\pi, \pi]]_{\varphi^! TM} = 0$ and $Ad_{\varphi^*}(\pi) = \pi$.
In the before example, a Hom-Poisson manifold (M, φ, π) is equipped with a Hom-Poisson tensor π.

Lemma

With the above notation,the following statements are equivalent:

$$\varphi^*(\{f,g\}) = \{\varphi^*(f), \varphi^*(g)\}, \quad Ad_{\varphi^*}(\pi) = \pi, \quad Ad_{\varphi^*} \circ \pi^\sharp = \pi^\sharp \circ Ad^\dagger_{\varphi^*} \tag{3}$$

where $Ad^\dagger_{\varphi^*} : \Gamma(\varphi^! T^*M) \to \Gamma(\varphi^! T^*M)$ *is given by*
$Ad^\dagger_{\varphi^*}(\xi)(X) = (\varphi^*\xi)(Ad^{-1}_{\varphi^*}(X))$ *and* $\pi^\sharp : \varphi^! T^*M \to \varphi^! T^*M$ *is a bundle map defined by*
$\pi^\sharp(\xi)(\eta) = \pi(\xi, \eta), \ \forall X \in \Gamma(\varphi^! TM), \ \forall \xi, \eta \in \Gamma(\varphi^! T^*M).$

Theorem

$(C^\infty(M), \{.,.\}, \varphi^*)$ *is a purely Hom-Poisson algebra iff* (M, φ, π) *is a Hom-Poisson manifold.*

Lemma

Let (M, φ, π) be a Hom-Poisson manifold. Define a bracket on $\Gamma(\varphi^! T^*M)$ for all $\xi, \eta \in \Gamma(\varphi^! T^*M)$ by

$$[\xi, \eta]_{\pi^\sharp} = L_{\pi^\sharp(\xi)}\eta - L_{\pi^\sharp(\eta)}\xi - d\pi(\xi, \eta), \tag{4}$$

in which the differential operator d is defined on $\Gamma(\wedge^\bullet T^*M)$. Given $\Phi : \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet T^*M)$, for all $X \in \Gamma(\wedge^k T^*M)$ and $\Xi \in \Gamma(\wedge^k TM)$, define $\Phi^\dagger : \Gamma(\wedge^\bullet TM) \to \Gamma(\wedge^\bullet TM)$ by $(\Phi^\dagger(X))(\Xi) = \varphi^* X(\Phi^{-1}(\Xi))$. Then,

$$d \circ \Phi^\dagger = \Phi^\dagger \circ d, \quad \Phi^\dagger(L_\Xi X) = L_{\Phi(\Xi)}\Phi^\dagger(X),$$

$$(\Phi^\dagger)^{-1} \circ L_\Xi = L_{\Phi^{-1}(\Xi)} \circ (\Phi^\dagger)^{-1}, \quad d \circ L_\Xi = -(-1)^k L_{\Phi(\Xi)} \circ d \tag{5}$$

Theorem

Let (M, φ, π) is a Hom-Poisson manifold. Then, $(\varphi^! T^*M, \varphi, Ad^\dagger_{\varphi^*}, [.,.]_{\pi^\sharp}, \pi^\sharp)$ is the cotangent Hom-Lie algebroid where $[.,.]_{\pi^\sharp}$ is given by the equation (4).

- L. Cai, J. Liu and Y. Sheng, *Hom-Lie algebroids, Hom-Lie bialgebroids and Hom-Courant algebroids*, arXiv: 1605.04752 v1, (2016).
- J. Hartwig, D. Larsson and S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra, 295 (2006), no. 2, pp. 314–361.
- C. Laurent-Gengoux and J. Teles, *Hom-Lie algebroids*, J. Geom. Phys., 68 (2013), pp. 69–75.
- A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl., 2 (2008), no. 2, pp. 51–64 .
- Y. Sheng, *Representation of Hom-Lie algebras*, Alg. Rep. Theory, 15 (2012), no. 6, pp. 1081–1098.