(Non)Linear Connections on Lie Algebroid Tangent Bundles

H. Arianpoor

Tafresh University
Department of Mathematics

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Kharazmi University

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Introduction

- Lie algebroids as a natural generalization of vector fields on a manifold has been introduced by Pradines.
- Lie algebroids (not necessarily as Lie algebra bundles) are a generalization of Lie algebra and vector bundles. They are vector bundles with an anchor map and a Lie bracket defined on the modules of sections induced from tangent bundle.
- Lie algebroids provide a natural setting in which one can develop the theory of differential operators (such as the exterior derivative and the Lie derivative) more general than that of the tangent and cotangent bundles of a smooth manifold and their exterior powers.
- They represent a widely domain of research, with applications in many areas of mathematics and physics.

Overview of paper topic

This paper is devoted to study in the geometry of the tangent bundle of a Lie algebroid. One approach is of the tangent bundle which has a natural structure of Lie algebroid, linearized by using the concept of linear connections. We give the procedure of deriving a (non)linear connection in respect to its local frames to obtain its correspondent on a Lie algebroid from the tangent bundle over the base manifold.

Definition

A **Lie algebroid** on a smooth n-manifold M is a smooth vector bundle E of rank m over M together with a bracket $[.,.]_E$ on the space of its smooth sections and a vector bundle morphism (the anchor) $\rho : E \to TM$ from smooth sections of the bundle E into smooth sections of the tangent bundle TM. It has equipped with the Lie algebra structure from the Lie bracket on TM via a Lie algebra homomorphism ρ_E between modules of smooth sections $\Gamma(E)$ and $\Gamma(TM)$ by the following

$$\rho_E([s_1, s_2]_E) = [\rho_E(s_1), \rho_E(s_2)]_{TM},$$
$$\rho_E([s_1, fs_2]_E) = f[\rho_E(s_1), \rho_E(s_2)]_{TM} + \rho_E(s_1)(f)\rho_E(s_2),$$

for all $s_1, s_2 \in \Gamma(E)$ and $f \in C^\infty(M)$.

The Lie bracket $[.,.]_E$ satisfies the Jacobi identity:

$$[s_1,[s_2,s_3]_E]_E+[s_2,[s_3,s_1]_E]_E+[s_3,[s_1,s_2]_E]_E=0.$$

Example

• Every Lie algebra is a Lie algebroid over a point. Two main Lie algebroids are of course offered by:
• The tangent bundle $TM \to M$ over a smooth manifold M with the usual bracket $[.,.]$ of vector fields and anchor map that is the identity $TM \to TM$.
• The cotangent bundle $T^*M \to M$ with the Poisson bracket $\{.,.\}$ on one forms and the anchor map $\rho : T^*M \to TM$ which is the Poisson tensor on manifold M.

For a local coordinate system $x = (x^k)_{1\leq k\leq n}$ on $U \subset M$ and a local frame $e = \{e_\alpha\}_{1\leq\alpha\leq m}$ of sections of E on U, we have local coordinates (x^k, u^α) on $\pi^{-1}(U) \subset E$, where $e = u^\alpha e_\alpha(x),\ e \in E$. Let $g_{UV} : U \cap V \to Gl(m, \mathbb{R})$ be the transition functions of E. The matrix of functions $(M^\alpha_\beta(x))$ represents $g_{UV}(x)$ in each $x \in U \cap V$. Then local coordinates $(\tilde{x}^k, \tilde{u}^\alpha)$ on $\pi^{-1}(V)$ change by the relations

$$\tilde{x}^k = \tilde{x}^k(x), \quad \tilde{u}^\alpha = M^\alpha_\beta(x)u^\beta. \tag{1}$$

The Jacobi matrix of the corresponding transformation is given by:

$$\begin{pmatrix} \frac{\partial \tilde{x}^k}{\partial x^h} & 0 \\ \\ \frac{\partial M^\alpha_\beta}{\partial x^h} u^\beta & M^\alpha_\beta \end{pmatrix} \tag{2}$$

Using the inverse matrix (R^β_α) of (M^α_β) and a basis $\{e_\alpha\}$ for the module $\Gamma(E)$, we have $\tilde{e}_\alpha = R^\beta_\alpha e_\beta$.

The action of the anchor map ρ and the Lie bracket $[.,.]_E$ is locally defined by the **structure functions** of the Lie algebroid E:

$$\rho(e_\alpha) = \rho_\alpha^k \frac{\partial}{\partial x^k}, \quad [e_\alpha, e_\beta]_E = C_{\alpha\beta}^\gamma e_\gamma, \tag{3}$$

where these functions $\rho_\alpha^k = \rho_\alpha^k(x)$ and $C_{\alpha\beta}^\gamma = C_{\alpha\beta}^\gamma(x)$ are given on M. A change of local chart on E implies $\tilde{\rho}_\alpha^k = R_\alpha^\beta \rho_\beta^h \frac{\partial \tilde{x}^k}{\partial x^h}$.

Theorem

The stucture functions of the Lie algebroid E satisfy the relations:

$$\rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial x^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial x^j} = \rho_\gamma^i C_{\alpha\beta}^\gamma.$$

For a local frame $\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial u^\alpha}\}$ on the tangent bundle TE in a fixed point, the jacobi matrix (2) changes by the rules:

$$\frac{\partial}{\partial x^h} = \frac{\partial \tilde{x}^k}{\partial x^h}\frac{\partial}{\partial \tilde{x}^k} + \frac{\partial M^\alpha_\beta}{\partial x^h} u^\beta \frac{\partial}{\partial \tilde{u}^\alpha}, \quad \frac{\partial}{\partial u^\beta} = M^\alpha_\beta \frac{\partial}{\partial \tilde{u}^\alpha}. \tag{4}$$

Theorem

The anchor ρ maps the local coordinates (x^k, u^α) on E, with the changes (1) to a local coordinates (x^k, η^k) on the tangent bundle TM, with changes $x'^k = x'^k(x)$, $\quad \eta'^k = \frac{\partial x'^k}{\partial x^h}\eta^h = u^\gamma \rho^h_\gamma \frac{\partial \tilde{x}^k}{\partial x^h}$.

The corresponding transformation implies the Jacobi matrix:

$$\begin{pmatrix} \frac{\partial x'^k}{\partial x^j} & 0 \\ \frac{\partial}{\partial x^j}(\rho^h_\gamma \frac{\partial \tilde{x}^k}{\partial x^h}) u^\gamma & \rho^h_\beta \frac{\partial \tilde{x}^k}{\partial x^h} \end{pmatrix} \tag{5}$$

The anchor ρ maps the local coordinates (x^k, u^α) on E to the local frame $(x^k, u^\alpha \rho_\alpha^k)$ on $\rho(E) \subset TM$. The Jacobi matrix of ρ is

$$\begin{pmatrix} \delta_h^k & 0 \\ \\ \frac{\partial \rho_\alpha^k}{\partial x^h} u^\gamma & \rho_\alpha^h \end{pmatrix} \tag{6}$$

The tangent mapping ρ_* is locally defined on $\rho(E) \subset TM$ by

$$\rho_*(\frac{\partial}{\partial x^k}) = \frac{\partial^*}{\partial x^k} = \frac{\partial}{\partial x^k} + u^\alpha \frac{\partial \rho_\alpha^h}{\partial x^k} \frac{\partial}{\partial \eta^h}, \quad \rho_*(\frac{\partial}{\partial u^\alpha}) = \frac{\partial^*}{\partial u^\alpha} = \rho_\alpha^h \frac{\partial}{\partial \eta^h} \tag{7}$$

The dual basis of the frame $\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial \eta^k}\}$ on $T(TM)$ induced by ρ_* is

$$d^* x^k = dx^k, \quad d^* \eta^k = u^\alpha \frac{\partial \rho_\alpha^k}{\partial x^h} dx^h + \rho_\alpha^k du^\alpha \tag{8}$$

For a change of local charts on $T(TM)$, the change laws on the tangent bundle TE are, due to the Jacobi matrix (5),

$$\frac{\partial^*}{\partial x^k} = \frac{\partial x'^k}{\partial x^j}\frac{\partial}{\partial x'^k} + \frac{\partial}{\partial x^j}(\rho^h_\gamma \frac{\partial x'^k}{\partial x^h})u^\gamma \frac{\partial}{\partial \eta'^k}, \quad \frac{\partial^*}{\partial u^\beta} = \rho^h_\beta \frac{\partial x'^k}{\partial x^h}\frac{\partial}{\partial \eta'^k} \quad (9)$$

Theorem

The tangent bundle TE is a Lie algebroid on the base manifold M with an anchor map $\varrho : TE \to TM$ defined as $\varrho = p_* \circ \rho_*$, where $p_* : T(TM) \to TM$ is the tangent map of the projection $p : TM \to M$, acting by $p_*(X^k \frac{\partial}{\partial x^k} + U^k \frac{\partial}{\partial \eta^k}) = X^k \frac{\partial}{\partial x^k}$ at a point (x, η) such that the following diagram commutes:

$$\begin{array}{ccc} TE & \xrightarrow{\rho_*} & T(TM) \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle p_*} \\ E & \xrightarrow{\rho} & TM \end{array}$$

Let $\pi_* : TE \to TM$ be the tangent mapping of the projection $\pi : E \to M$. The **vertical bundle** VE of E is given by $VE = \ker \pi_*$. A local frame on VE is $\{\frac{\partial}{\partial u^\alpha}\}_{1\leq\alpha\leq m}$ and if $\pi^*(TM)$ is the pull back of the tangent bundle TM, then the splitting sequence is obtained:

$$0 \longrightarrow VE \xrightarrow{i} TE \xrightarrow{d\pi} \pi^*(TM) \longrightarrow 0\,. \qquad (10)$$

Definition (nonlinear connection on TE)

The splitting $C : TE \to VE$ in sequence (10) is called a **connection** on the vertical bundle. It determines the decomposition $TE = VE \bigoplus HE$ where HE is the **horizontal subbundle** of E, isomorphic to the pull back bundle $\pi^*(TM)$ by the differential $d\pi$. This isomorphism is called **nonlinear connection** on the Lie algebroid E.

The **horizontal lift** $\varpi : \pi^*(TM) \to HE$ associated to the nonlinear connection is defined by

$$\varpi(\frac{\partial}{\partial x^k}) = \frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N_k^\alpha \frac{\partial}{\partial u^\alpha}, \tag{11}$$

where N_k^α are the *coefficients* of the nonlinear connection on E.
A change of local charts implies that $\frac{\delta}{\delta x^h} = \frac{\partial \tilde{x}^k}{\delta x^h} \frac{\delta}{\delta \tilde{x}^k}$. From relations (4), the laws of change for N_k^α are obtained

$$\frac{\partial \tilde{x}^k}{\partial x^h} \tilde{N}_k^\alpha = M_\beta^\alpha N_h^\beta - \frac{\partial M_\beta^\alpha}{\partial x^h} u^\beta. \tag{12}$$

Theorem

The Lie brackets of the local frame $\{\frac{\delta}{\delta x^k}, \frac{\partial}{\partial u^\alpha}\}$ *on* TE *are*

$$[\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^h}] = (\frac{\partial N_k^\alpha}{\partial x^h} - \frac{\partial N_h^\alpha}{\partial x^k}) \frac{\partial}{\partial u^\alpha}, \quad [\frac{\delta}{\delta x^k}, \frac{\partial}{\partial u^\beta}] = [\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}] = 0.$$

For the local frame $\{\frac{\delta}{\delta x^k}, \frac{\partial}{\partial \eta^k}\}$ of a nolinear connection on $T(TM)$ with

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N_k^h \frac{\partial}{\partial \eta^h} \tag{13}$$

and the dual basis $\{dx^k, \delta\eta^k\}$ with $\delta\eta^k = d\eta^k + N_h^k dx^h$, then the coefficients of the nonlinear connection change by the rules:

$$N_h'^j \frac{\partial x'^h}{\partial x^k} = \frac{\partial x'^j}{\partial x^h} N_k^h - \frac{\partial^2 x'^j}{\partial x^h \partial x^k} \eta^h. \tag{14}$$

So any section of $\Gamma(TE)$ is decomposed in the local frame $\{\frac{\delta}{\delta x^k}, \frac{\partial}{\partial u^\alpha}\}$ of a nonlinear connection.
In the dual basis $\{dx^k, \delta u^\alpha\}$ of the local frame on TE we have $\delta u^\alpha = du^\alpha + N_h^\alpha dx^h$.

Definition (linear connection on TE)

A linear map $\nabla : \Gamma(TE) \times \Gamma(TE) \to \Gamma(TE)$, $(X, Y) \mapsto \nabla_X Y$ is called **linear connection** on TE such that satisfies the following

$$\nabla_X(fY) = (\varrho(X)f)Y + f\nabla_X Y, \quad \forall f \in C^\infty(E),\ \forall X, Y \in \Gamma(TE) \tag{15}$$

Theorem

If the linear connection ∇ on TE preserves the bundle HE, then it has the following coefficients:

$$\nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \nabla^i_{jk} \frac{\delta}{\delta x^i}, \quad \nabla_{\frac{\partial}{\partial u^\gamma}} \frac{\delta}{\delta x^j} = \nabla^i_{j\gamma} \frac{\delta}{\delta x^i},$$
$$\nabla_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial u^\beta} = \nabla^\alpha_{\beta k} \frac{\partial}{\partial u^\alpha}, \quad \nabla_{\frac{\partial}{\partial u^\gamma}} \frac{\partial}{\partial u^\beta} = C^\alpha_{\beta\gamma} \frac{\partial}{\partial u^\alpha}$$

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