Integrable Dirac Structures on Lie Algebroids

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Introduction

- The theory of Dirac structures on vector spaces and their extension to manifolds was first introduced as a generalization of symplectic and Poisson structures by Courant, Weinstein and Dorfman.
- A Dirac structure on a manifold M is a smooth subbundle of the Whitney sum bundle $TM \oplus T^*M$, satisfying maximally isotropic property under a symmetric bilinear form. The concept was generalized to similar subbundles defined on the Whitney sum of the form $A \oplus A^*$ where (A, A^*) is a Lie bialgebroid.
- Lie algebroids as a natural generalization of vector fields on a manifold has been used in the algebraic geometric framework by Pradines.
- From a geometrical point of view, Dirac structures are closely related to Lie algebroids and Lie bialgebroids.

Introduction

- The notion of integrability of Dirac structures was defined first by Courant and after that by Fernandez. It leads to a poisson algebra of functions on the manifold making it possible to construct the classical mechanics on manifolds.
- A generalized definition of Dirac structures on Lie algebroids concludes to the specification of integrable Dirac structures.
- The notion of the characteristic pair of a Lie algebroid yields specific conditions on the closedness of generalized Dirac structures.
- The equivalence class of characteristic pairs corresponds to a generalized Dirac structure for the Lie bialgebroid case.
- When the generalized Dirac structure is integrable, the set of characteristic pairs defines a closed Lie algebra structure on any maximally isotropic subbundle of the Lie bialgebroid.

Overview of paper topic

In this paper we describe the integrability of Dirac structures on a given Lie bialgebroid by corresponding characteristic pairs. It is based on a closedness condition of the Courant bracket applied to sections of the Dirac structure. The characterization of integrable Dirac structures on a Lie algebroid can be done in terms of its subbundles and suitable tensors. This generalizes the concept of Dirac structures defined on the tangent bundle of a manifold.

Definition

A **Lie algebroid** on a smooth manifold M is a vector bundle $A \to M$ together with a bracket $[.,.]_A : \Gamma^\infty(A) \to \Gamma^\infty(A)$ on the space of its smooth sections and a bundle homomorphism (the anchor) $\rho : \Gamma^\infty(A) \to \Gamma^\infty(M)$ from smooth sections of the bundle A into smooth sections of the bundle TM, equipped with the natural Lie algebra structure $\Gamma^\infty(M)$ such that the following condition holds:

$$\rho([X, Y]_A) = [\rho(X), \rho(Y)] \tag{1}$$

and the Lie algebroid bracket satisfies the Leibniz rule for the module of sections over $C^\infty(M)$:

$$[X, fY]_A = f[X, Y]_A + (\rho(X)f)Y. \tag{2}$$

for all $X, Y \in \Gamma^\infty(A)$ and $f \in C^\infty(M)$.

Let $A^\star$ be the dual bundle of A (of rank k) over the manifold M and denote by $\bigwedge^p A$ the p-th external power of the bundle A as a vector bundle whose fibres are vector spaces of p-multilinear skew-symmetric forms on the dual space $A^\star$. Similarly, it will be denoted by $\bigwedge^p A^\star$ the p-th external power of the dual bundle $A^\star$.

Denote by $\mathcal{A}^p(A)$ the space of smooth sections of the bundle $\bigwedge^p A$ and by $\Omega^p(A)$ the space of smooth sections of the dual bundle $\bigwedge^p A^\star$. The direct sums $\mathcal{A}(A) = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}^p(A)$ and $\Omega(A) = \bigoplus_{p\in\mathbb{Z}} \Omega^p(A)$, are taken for all integers p which satisfy $0 \leq p \leq k$, for $p = 0$, $\mathcal{A}^0(A)$ and $\Omega^0(A)$ both coincide with the space $C^\infty(M)$.

Operations such as the interior product, the exterior product and pairing can be defined in the $\mathbb{Z}$-graded vector spaces $\mathcal{A}(A)$ and $\Omega(A)$ as extention of these notions in $\bigwedge A$ and $\bigwedge A^\star$.

Differential of Lie algebroid

The $\mathbb{Z}$-garded space $\Omega(A)$ is equipped with the natural differential $d_A : \Omega^p(A) \to \Omega^{p+1}(A)$ defined on any $\omega \in \Omega^p(A)$ by

$$(d_A\omega)(X_1, ..., X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \rho(X_i)\omega(X_1, ..., \check{X}_i, ..., X_{p+1}) + \sum_{1 \le i < j \le p+1} (-1)^{i+j} \omega([X_i, X_j]_A, X_1, ..., \check{X}_i, ..., \check{X}_j, ..., X_{p+1}) \quad (3)$$

where $\check{X}_i$ means omission of the argument.
For $f \in C^\infty(M)$ we have $\langle d_A f, X\rangle = \rho(X)f, \ \forall X \in \Gamma^\infty(A)$.

The differential d_A for Lie algebroid A can be related to the de Rham differential of the manifold M as $d_A = \rho^* \circ d$, using by $\rho^\star$ the transpose of the anchor ρ.

Lie derivative on Lie algebroid

The Lie derivative $\mathcal{L}_X^A$ with respect to each smooth section $X \in \Gamma^\infty(A)$ as a graded endomorphism of degree 0 of the graded algebra $\Omega(A)$, can be defined by generalizing Cartan's formula

$$\mathcal{L}_X^A = i_X \circ d_A + d_A \circ i_X \tag{4}$$

where the interior product i_X is a derivation of degree -1 of the algebra $\Omega(A)$, as the $(q-1)$-multilinear form on $\mathcal{A}^1(A) = \Gamma^\infty(A)$ defined by $i_X\omega(X_1, ..., X_{q-1}) = \omega(X, X_1, ..., X_{q-1})$, for all $X_1, ..., X_{q-1} \in \Gamma^\infty(A)$ and $\omega \in \Omega^q(A)$.
For $f \in C^\infty(M)$ we see that $\mathcal{L}_X^A f = i_{\rho(X)} df$.

Corollary

For any $\alpha \in \Omega^1(A)$ *we have* $\langle Y, \mathcal{L}_X^A \alpha \rangle = \rho(X)\langle Y, \alpha\rangle - \alpha([X, Y]_A)$.

Lemma

Let A be a Lie algebroid over a smooth manifold M. Then, for any $X, Y \in \Gamma^\infty(A)$ and $f \in C^\infty(M)$, the Lie derivative has the following properties:

$$\mathcal{L}_X^A(d_A f) = d_A(\mathcal{L}_X^A f), \quad i_{[X,Y]_A} = [\mathcal{L}_X^A, i_Y]_A, \quad \mathcal{L}_{[X,Y]_A}^A = [\mathcal{L}_X^A, \mathcal{L}_Y^A]_A$$

Definition

Let V be a vector space and consider also its dual space $V^\star$ with respect to the dual inner product $\langle ., . \rangle$. Let a bilinear symmetric form $\langle ., . \rangle_+$ defined on $V \times V^\star$ as
$\langle (x, y), (x', y') \rangle_+ = \langle x, y' \rangle + \langle x', y \rangle$ for $x, x' \in V$ and $y, y' \in V^\star$.
A **Dirac structure** on V is a subspace $D \subset V \oplus V^\star$ which is maximally isotropic under the pairing $\langle ., . \rangle_+$ such that $D^\perp = D$.

Example

Let $T : V \to V^\star$ be a skew symmetric linear map. Then $graph(T) \subset V \oplus V^\star$ is maximally isotropic under $\langle .,. \rangle_+$ for which $\langle Tx, x' \rangle + \langle Tx', x \rangle = 0$ so $graph(T)$ is a Dirac structure on V.

On a vector space V, there is the natural projection $\pi : V \oplus V^\star \to V$ that gives rise to the characterization equation $\pi(D)^\circ = D \cap V^\star$ of Dirac structure D where for a vector space W, W° is the annihilator of W.

Theorem

A Dirac structure $D \subset V \oplus V^\star$ induces a skew form on the subspace $\pi(D) \subset V$ with the kernel $D \cap V$. Consequently, the Dirac structure induces a skew bivector on the quotient space $V/D \cap V$.

Definition

A **generalized Dirac structure** on a smooth manifold M is a subbundle $D \subset TM \oplus T^{\star}M$ which is maximally isotropic under the symmetric pairing $\langle (X,\alpha),(Y,\beta)\rangle_{+} = i_{Y}\alpha + i_{X}\beta) = \alpha(Y) + \beta(X)$ for $X, Y \in TM$ and $\alpha, \beta \in T^{\star}M$.

To define a closed Dirac structure on M it is necessary to give a skew symmetric bracket $[.,.]$ on $D \subset TM \oplus T^{\star}M$ such that for any two sections $\sigma_1, \sigma_2 \in \Gamma^{\infty}(D)$ we have $[\sigma_1, \sigma_2] \in \Gamma^{\infty}(D)$.

Definition

A generalized Dirac structure on M is called **closed** if for any three sections $\sigma_i = (X_i, \alpha_i),\ i = 1, 2, 3$ the following property holds:

$$\langle X_1, \mathcal{L}_{X_2}\alpha_3\rangle + \langle X_3, \mathcal{L}_{X_1}\alpha_2\rangle + \langle X_2, \mathcal{L}_{X_3}\alpha_1\rangle = 0 \tag{5}$$

A closed generalized Dirac structure $D \subset TM \oplus T^{\star}M$ yields a Lie algebroid structure on it which is due to Courant.

Theorem (Courant)

A generalized Dirac structure D on the manifold M is closed iff it is a Lie algebroid with the anchor $\rho : D \to TM$ and the Lie algebra structure on the space of sections is defined as:

$$[(X_1, \alpha_1), (X_2, \alpha_2)] = ([X_1, X_2], \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 - \frac{1}{2} d \circ (i_{X_1}\alpha_2 - i_{X_2}\alpha_1)). \quad (6)$$

Example

Given a Poisson manifold (M, J), the Poisson tensor J defines a mapping: $\hat{J} : T^{\star}M \to TM, \quad \hat{J}(df)(dg) = J(df, dg) = \{f, g\}$.
The subbundle D defined by the gragh of the mapping $\hat{J}$ induces the generalized Dirac structure $D = \{(\hat{J}(\alpha), \alpha) | \alpha \in T^{\star}M\}$ on M.

A Lie algebroid structure for the dual bundle $A^\star$ of a Lie algebroid A defines by a Lie algebra structure $[.,.]_{A^\star}$ and an anchor $\rho^\star : \Gamma^\infty(A^\star) \to \Gamma^\infty(M)$ which satisfies the conditions (1) and (11). The differential $d_{A^\star}$ acts on the space of smooth sections of $A^\star$.

Definition

A pair of Lie algebroids $(A, A^\star)$ is said to be a **Lie bialgebroid** if the differentials d_A and $d_{A^\star}$ are as derivations of Schouten bracket of $A^\star$ and for the commutator of the smooth sections of A:

$$d_A[\alpha_1, \alpha_2] = [d_A\alpha_1, \alpha_2] + [\alpha_1, d_A\alpha_2], \ \forall \alpha_1, \alpha_2 \in \Gamma^\infty(A^\star) \quad (7)$$

$$d_A^\star[X_1, X_2] = [d_A^\star X_1, X_2] + [X_1, d_A^\star X_2], \ \forall X_1, X_2 \in \Gamma^\infty(A) \quad (8)$$

Example

The trivial Lie algebroid structure of TM and $T^\star M$ with the null anchor, leads to that $TM \oplus T^\star M$ is a Lie bialgebroid.

Definition

Let $(A, A^\star)$ be a Lie bialgebroid and consider the Whitney sum $B = A \oplus A^\star$. A subbundle $D \subset A \oplus A^\star$ is called a **generalized Dirac structure** on M if it is maximally isotropic with respect to the symmetric canonical form $\langle ., . \rangle_+ : B \times B \to B$, can be defined by the duality between the two bundles A and $A^\star$ as follows:

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle_+ = \langle X_1, \alpha_2 \rangle + \langle X_2, \alpha_1 \rangle, \ \forall (X_1, \alpha_1), (X_2, \alpha_2) \in B \tag{9}$$

The Lie algebra structure on $\Gamma^\infty(D)$, the space of smooth sections of D needs to both of Lie algebroid structures on A and $A^\star$.

Definition

Let $(A, A^\star)$ be a Lie bialgebroid. Ther exists a skew symmetric bilinear operation on the sections of $B = A \oplus A^\star$ in the form:

$$[(X_1, \alpha_1), (X_2, \alpha_2)]_B = ([X_1, X_2]_A + [X_1, X_2]_{\mathcal{L}^{A^\star}}, [\alpha_1, \alpha_2]_A + [\alpha_1, \alpha_2]_{\mathcal{L}^A}) \tag{10}$$

where $[X_1, X_2]_{\mathcal{L}^{A^\star}} = \mathcal{L}^{A^\star}_{\alpha_1} X_2 - \mathcal{L}^{A^\star}_{\alpha_2} X_1 - \frac{1}{2} d_{A^\star} \circ (i_{X_1}\alpha_2 - i_{X_2}\alpha_1)$ and $[\alpha_1, \alpha_2]_{\mathcal{L}^A} = \mathcal{L}^A_{X_1}\alpha_2 - \mathcal{L}^A_{X_2}\alpha_1 - \frac{1}{2} d_A \circ (i_{X_1}\alpha_2 - i_{X_2}\alpha_1)$.

Definition

The subbundle $D \subset B$ is said to be a generalized Dirac structure if the operation above induces a Lie algebra structure on $\Gamma^\infty(D)$.

Theorem

Let $(A, A^\star)$ be a Lie bialgebroid. Consider also the operation $[.,.]_B$ and the anchor $\rho_B = \rho \oplus \rho^\star : \Gamma^\infty(B) \to \Gamma^\infty(M)$ for $B = A \oplus A^\star$. A subbundle $D \subset B$ is a generalized Dirac structure iff $(D, [.,.]_D, \rho_D)$ is a Lie algebroid in which $[.,.]_B$ and ρ_B restrict to the bundle D.

The characterization of Dirac structures can be done in terms of subbundles I of A and A-tensors Ω, generalizes Dirac structures on $TM \oplus T^\star M$.

Definition

Let $(A, A^\star)$ be a Lie bialgebroid. The **characteristic pair** of the Dirac structure D is a pair (I, Ω) of a smooth subbundle $I \subset A$ and a bivector $\Omega \in \Gamma^\infty(\bigwedge^2 A)$ associated to a maximally isotropic subbundle of $A \oplus A^\star$ under the symmetric pairing (9), corresponds to the Dirac structure $D = \{(X + \Omega^\sharp \alpha, \alpha) | \forall X \in I,\ \alpha \in I^\perp\}$, where $I^\perp$ is the co-normal bundle of I in $A^\star$.

Lemma

For given a Dirac structure $D \subset A \oplus A^\star$ and a subbundle $I \subset D$ then there exists the bundle map $\Omega^\sharp$ restricted to $I^\perp$ which is equivalent to a bivector field on the quotient bundle A/I.

Corollary

Two characteristic pairs (I_1, Ω_1), (I_2, Ω_2) are equivalent iff $I_1 = I_2 = I$ and $\Omega_1^\sharp(\alpha) - \Omega_2^\sharp(\alpha) \in I, \ \forall \alpha \in I^\perp$.
This leads to the equivalence of the equivalent classes with the set of generalized Dirac structures of a given Lie bialgebroid.

The characteristic pair is merely associated to the existence of a maximally isotropic subbundle of the Lie bialgebroid with respect to the pairing (9) without the Lie algebra structure (10) restricted to the subbundle is to be closed.

Theorem

Let $(A, A^\star)$ be a Lie bialgebroid and D a subbundle of $A \oplus A^\star$, maximally isotropic under the pairing (9) and corresponding to the characteristic pair (I, Ω). Then, D is Dirac structure iff I is a Lie subalgebroid and the Schouten bracket of $A^\star$ is closed on $\Gamma^\infty(I^\perp)$:

$$[\alpha, \beta] = [\alpha, \beta]_{A^\star} + \mathcal{L}^A_{\Omega^\sharp \alpha}\beta - \mathcal{L}^A_{\Omega^\sharp \beta}\alpha - d_A(\Omega(\alpha, \beta)), \quad \forall \alpha, \beta \in \Gamma^\infty(I^\perp) \tag{11}$$

where $[.,.]$ stands for the Schouten bracket of A.

Corollary

A subbundle $D \subset TM \oplus T^\star M$ corresponding to the characteristic pair (I, Ω) is a generalized Dirac structure on M iff I is a Lie subalgebroid and Ω defines a Poisson structure $[\Omega, \Omega] = 0$ on the quotient space $\Omega^\sharp(\Gamma^\infty(I^\perp)/I$.

Lemma

Let D be a generalized Dirac structure with the characteristic pair (I, Ω). Then, $\mathcal{L}^A_X \alpha \in I^{\perp}, \quad \forall X \in I,\ \alpha \in I^{\perp}$.

Theorem

Let $(A, A^{\star})$ be a Lie bialgebroid and consider also a generalized Dirac structure represented by the equivalence class of characteristic pair $[(I, \Omega)]$. Then all other representations satisfy the conditions of the previous theorem.

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