$\star$-CALCULUS ON QUANTIZED COORDINATE SPACE

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Introduction

- The study of a classical and physical system is by investigating certain conditions on the algebra of observables which encodes so much imformation about it.
- Differential calculus over the commutative function algebra which is closely related to that physical system.
- The noncommutative version of the above theory is the algebraic and geometric approach to noncommutative function algebras.
- One approach in this study is the quantization of coordinate space using quantum groups.
- An interpretation of the quantum space can develope the appropriate theories on the quantized coordinate spaces parallel to those on the classical ones.

Overview of paper topic

In this paper we deal with the deformation quantization of the coordinate algebra of the quantum N-space and give the construction of first order $\star$-differential calculus which is based on the formalism of $\star$-product on a commutative algebra making it into a noncommutative space. This formalism plays an important role in developing fundamental concepts of the classical differential calculus to the noncommutative generalization of the notion of q^2-differentiations on the quantum N-space and also provides useful tools for the study of its geometry.

Let $\mathbb{C}\langle z_1, ..., z_N\rangle$ be the associative, commutative, unital free $\mathbb{C}$-algebra of formal power series in N coordinates $z_1, ..., z_N$. The quantum N-sapce is the quadratic algebra $\mathbb{C}_q^N = \frac{\mathbb{C}\langle z_1,...,z_N\rangle}{\mathcal{R}}$ where $\mathcal{R}$ is the ideal generated by the relations $z_i z_j - q z_j z_i, \ \ i < j$ and $0 \neq q \in \mathbb{C}$. Functions on this space are considered to be formal power series in $z_1, ..., z_N$. Set $\hat{z}_i = z_i$ (modulo $\mathcal{R}$), $i = 1, ..., N$. The induced relations on $\mathbb{C}_q^N$ are satisfied $\hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \quad i < j$.

In general for the monomials,

$$(\hat{z}_i)^{m_i}(\hat{z}_j)^{m_j} = q^{m_i m_j}(\hat{z}_j)^{m_j}(\hat{z}_i)^{m_i}, \quad i < j$$

$$(\hat{z}_1)^{l_1}...(\hat{z}_N)^{l_N}.(\hat{z}_1)^{k_1}...(\hat{z}_N)^{k_N} = q^{(-\sum_{i=1}^{N-1} k_i \sum_{j=i+1}^{N} l_j)}(\hat{z}_1)^{l_1+k_1}...(\hat{z}_N)^{l_N+k_N}$$

Let $\mathbb{C}\langle z_1, ..., z_N \rangle$ be the associative, commutative, unital free $\mathbb{C}$-algebra of formal power series in N coordinates $z_1, ..., z_N$. The quantum N-sapce is the quadratic algebra $\mathbb{C}_q^N = \frac{\mathbb{C}\langle z_1, ..., z_N \rangle}{\mathcal{R}}$ where $\mathcal{R}$ is the ideal generated by the relations $z_i z_j - q z_j z_i, \; i < j$ and $0 \neq q \in \mathbb{C}$. Functions on this space are considered to be formal power series in $z_1, ..., z_N$. Set $\hat{z}_i = z_i$ (modulo $\mathcal{R}$), $i = 1, ..., N$. The induced relations on $\mathbb{C}_q^N$ are satisfied $\hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \quad i < j$.

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$$(\hat{z}_1)^{l_1} ... (\hat{z}_N)^{l_N} . (\hat{z}_1)^{k_1} ... (\hat{z}_N)^{k_N} = q^{(-\sum_{i=1}^{N-1} k_i \sum_{j=i+1}^{N} l_j)} (\hat{z}_1)^{l_1 + k_1} ... (\hat{z}_N)^{l_N + k_N}$$

Let $\mathbb{C}[[h]]$ be the unital commutative algebra of formal power series in h, ($q = e^h$) with coefficients in $\mathbb{C}$. Addition and multiplication in $\mathbb{C}[[h]]$ are given formally by

$$a + b = \sum_{n=0}^{\infty}(a_n + b_n)h^n, \quad ab = \sum_{n=0}^{\infty}(\sum_{r=0}^{n} a_r b_{n-r})h^n$$

for $a = \sum_{n=0}^{\infty} a_n h^n$ and $b = \sum_{n=0}^{\infty} b_n h^n$.

h-Adic Topology

Topological tensor product of two algebras $\mathbb{C}\langle z_1, ..., z_N\rangle \hat{\bigotimes} \mathbb{C}[[h]]$ is $\mathbb{C}[[h]]$-isomorphic to $\mathbb{C}\langle z_1, ..., z_N\rangle[[h]]$, the algebra of formal power series in h with coefficients in $\mathbb{C}\langle z_1, ..., z_N\rangle$.

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$\mathbb{C}_q^N$ can be regarded as a $\mathbb{C}[[h]]$-algebra where the product of two functions on $\mathbb{C}_q^N$ will be a formal power series in $\hat{z}_1, ..., \hat{z}_N$ with coefficients in $\mathbb{C}[[h]]$.

Poincare-Birkhoff-Witt property

By considering the normal ordering of the coordinates $\hat{z}_1, ..., \hat{z}_N$ and given $\{(\hat{z}_1)^{i_1}...(\hat{z}_N)^{i_N} | i_1, ..., i_N \geq 0\}$ as a $\mathbb{C}$-vector space basis for $\mathbb{C}_q^N$, the dimension of the subspace spanned by monomials of a fixed degree in $\hat{z}_1, ..., \hat{z}_N$ is the same as the dimension of the subspace spanned by monomials in the commutative N variable.

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Let $\mathbb{C}_h\langle z\rangle$ be $\mathbb{C}[[h]]$-module of formal power series in $z_1, ..., z_N$ with values in $\mathbb{C}[[h]]$. An element of $\mathbb{C}_h\langle z\rangle$ is of the form $f = \sum_{i_1,...,i_N \geq 0} f_{i_1,...,i_N}(z_1)^{i_1}...(z_N)^{i_N}$ with coefficients in $\mathbb{C}[[h]]$.

Quantized N-Space

A $\mathbb{C}[[h]]$-module isomorphism $\mathbf{W} : \mathbb{C}_h\langle z\rangle \to \mathbb{C}_q^N$, defined on the generators by $\mathbf{W}((z_1)^{i_1}...(z_N)^{i_N}) = (\hat{z}_1)^{i_1}...(\hat{z}_N)^{i_N}$.
$\mathbf{W}$ can be extended to a $\mathbb{C}[[h]]$-algebra isomorphism by the following composition law on $\mathbb{C}_h\langle z\rangle$:

$$f \star g = \mathbf{W}^{-1}(\mathbf{W}(f).\mathbf{W}(g))$$

This relation makes $\mathbb{C}_h\langle z\rangle$ into an associative, noncommutative unital $\mathbb{C}[[h]]$-algebra. The image of $f \in \mathbb{C}_h\langle z\rangle$ under $\mathbf{W}$ is denoted by $\hat{f}$ which is $\hat{f} = \sum_{i_1,...,i_N \geq 0} f_{i_1,...,i_N}(\hat{z}_1)^{i_1}...(\hat{z}_N)^{i_N}$.

Let $\mathbb{C}_h\langle z\rangle$ be $\mathbb{C}[[h]]$-module of formal power series in $z_1,\ldots,z_N$ with values in $\mathbb{C}[[h]]$. An element of $\mathbb{C}_h\langle z\rangle$ is of the form $f=\sum_{i_1,\ldots,i_N\geq 0} f_{i_1,\ldots,i_N}(z_1)^{i_1}\ldots(z_N)^{i_N}$ with coefficients in $\mathbb{C}[[h]]$.

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Lemma

The ⋆-product satisfies the following commutation relation

$$f \star g = q^{\left(\sum_{r=1}^{N} k_{N-r+1}\left(\sum_{s=1}^{N-r} l_s - \sum_{t=N-r+2}^{N} l_t\right)\right)} g \star f$$

where $f = \sum f_{l_1,\dots,l_N}(z_1)^{l_1}\dots(z_N)^{l_N}, \;\; g = \sum g_{k_1,\dots,k_N}(z^1)^{k_1}\dots(z^N)^{k_N}$.

The left multiplication Z_i of functions by coordinates and the scaling operators U_a^i, $U_a^{i,j}$ for $i, j = 1, \dots, N$, are defined by

$$(Z_i f)(z) = z_i f(z), \quad f \in \mathbb{C}_h\langle z\rangle$$
$$(U_a^i f)(z) = f(q^a z_i)$$
$$(U_a^{i,j} f)(z) = U_a^i \dots U_a^j f(z), \quad i < j$$

For $i \geq j$ and $k < 1$, $U_a^{i,j}$ and U_a^k are defined to be identity.

Lemma

The $\star$-product satisfies the following commutation relation

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q-differentiations on $\mathbb{C}_h\langle z\rangle$

The q-differentiation operator $D^i_{q^a}$ $(a \in \mathbb{Z})$ is given by

$$D^i_{q^a} f(z) = \mathbf{W}^{-1}(D^i_{q^a}\hat{f}(\hat{z})) = \frac{f(z) - f(q^a z_i)}{(1-q^a)z_i}, \quad f \in \mathbb{C}_h\langle z\rangle$$

for $\hat{f} = \mathbf{W}(f)$ and $\hat{f}(q^a\hat{z}_i) = \hat{f}(\hat{z}_1, ..., \hat{z}_{i-1}, q^a\hat{z}_i, \hat{z}_{i+1}, ..., \hat{z}_N)$.
In the limit case $q \to 1$, $D^i_{q^a}$ tends to the usual partial derivative ∂_i.

Lemma

The q-differentiations, the left multiplications and the scaling operators satisfy the following equations

$$(i)\quad U^i_a(f\star g)=U^i_af\star U^i_ag,\quad U^{i,j}_a(f\star g)=(U^{i,j}_af)\star(U^{i,j}_ag)$$

$$(ii)\quad D^i_{q^a}Z_i-Z_iD^i_{q^a}=U^i_a$$

$$(iii)\quad U^i_bD^i_{q^a}=q^{-a}D^i_{q^a}U^i_b,\quad U^j_bD^i_{q^a}=D^i_{q^a}U^j_b,\quad i\neq j$$

$$(iv)\quad U^i_aZ_i=q^aZ_iU^i_a,\quad U^i_aZ_j=Z_jU^i_a,\quad i\neq j$$

$$(v)\quad D^i_{q^a}D^j_{q^b}=q^{-1}D^j_{q^b}D^i_{q^a},\quad i<j$$

Definition

The $\star$-derivative operators $\partial_i^\star$ on $\mathbb{C}_h\langle z\rangle$ are defined by

$$\partial_i^\star \triangleright f = D^i_{q^2} U_1^{1,i-1} U_2^{i+1,N} f, \quad i = 1, ..., N$$

Theorem

The $\star$-derivatives $\partial_i^\star$ satisfy the $\star$-deformed Leibniz rule:

$$\partial_i^\star \triangleright (f \star g) = (\partial_i^\star \triangleright f) \star (U_2^{i+1,N} g) + (U_1^{1,i-1} U_1^{i+1,N} U_2^i f) \star (\partial_i^\star \triangleright g)$$

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Definition

The $\star$-product between $\star$-derivatives $\partial_i^\star$, the scaling operators U_a^i and functions of $\mathbb{C}_h\langle z\rangle$ are defined by

$$
\begin{aligned}
&(\partial_i^\star \star f) \triangleright g = \partial_i^\star \triangleright (f \star g)\\
&(f \star \partial_i^\star) \triangleright g = f \star (\partial_i^\star \triangleright g)\\
&(U_a^i \star \partial_i^\star) \triangleright f = U_a^i(\partial_i^\star \triangleright f), \quad (\partial_i^\star \star U_a^i) \triangleright f = \partial_i^\star \triangleright (U_a^i f)\\
&(U_a^i \star U_b^j) \triangleright f = U_a^i U_b^j f, \quad (U_a^{i,j} \star U_b^{k,l}) \triangleright f = (U_a^{i,j} U_b^{k,l}) f\\
&(\partial_i^\star \star \partial_j^\star) \triangleright f = \partial_i^\star \triangleright (\partial_j^\star \triangleright f)\\
&(f \star U_a^i) \triangleright g = f \star U_a^i g, \quad (f \star U_a^{i,j}) \triangleright g = f \star U_a^{i,j} g\\
&(U_a^i \star f) \triangleright g = U_a^i(f \star g), \quad (U_a^{i,j} \star f) \triangleright g = U_a^{i,j}(f \star g)
\end{aligned}
$$

The ⋆-deformed Leibnitz rule and these equations holds:

$$\partial_i^\star \star z_j = q z_j \star \partial_i^\star, \quad j \neq i$$
$$\partial_i^\star \star z_i - q^2 z_i \star \partial_i^\star = U_2^{i+1,N}$$
$$U_a^i \star z_i = q^a z_i \star U_a^i, \quad U_a^i \star z_j = z_j \star U_a^i, \quad j \neq i$$
$$U_a^i \star \partial_i^\star = q^{-a} \partial_i^\star \star U_a^i$$
$$U_a^i \star U_b^j = U_b^j \star U_a^i$$
$$\partial_i^\star \star \partial_j^\star = q^{-1} \partial_j^\star \star \partial_i^\star, \quad i < j$$

Theorem

Let $(\mathcal{H}, \star)$ be the $\mathbb{C}[[h]]$-algebra with generators $\mathbf{1}$, the $\star$-derivatives $\partial_i^\star$'s, scaling operators U_a^i's, their inverse $(U_a^i)^{-1} = U_{-a}^i$ and functions on $\mathbb{C}_h\langle z\rangle$, with the $\star$-product structure of Definition (5). Let $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $\varepsilon : \mathcal{H} \to \mathbb{C}$ be defined on the generators by

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \quad \Delta(\partial_\rho^\star) = \partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star,$$

$$\Delta((U_a^\rho)^{\pm 1}) = (U_a^\rho)^{\pm 1} \otimes (U_a^\rho)^{\pm 1}, \quad \Delta(z^\rho) = z^\rho \otimes \mathbf{1} + \mathbf{1} \otimes z^\rho.$$

$$\varepsilon(\mathbf{1}) = \varepsilon(\partial_\rho^\star) = \varepsilon(z^\rho) = 0, \quad \varepsilon((U_a^\rho)^{\pm 1}) = 1.$$

Then $(\mathcal{H}, \star, \eta, \Delta, \varepsilon)$ is a bi-algebra with the unit map $\eta(1) = \mathbf{1}$.

Definition

Let H be a bi-algebra and $H_1, \ H_2$ its sub bi-algebras. A *left triple* (H_1, H_2, H)*-module algebra* is a unital $\mathbb{C}$-algebra M such that

- M is a left H_1-module algebra by the action of a $\mathbb{C}$-linear map $\alpha : H_1 \bigotimes M \to M$ satisfies $\alpha \circ (id_{H_1} \otimes \alpha) = \alpha \circ (._{H_1} \otimes id_M), \ \alpha(1 \otimes m) = m$
- M is a left H_2-module bi-algebra by the action of a $\mathbb{C}$-linear map $\beta : H_2 \bigotimes M \to M$ satisfies $\beta \circ (id_{H_2} \otimes \beta) = \beta \circ (m_{H_2} \otimes id_M), \ \beta(1 \otimes m) = m, \ \beta \circ (id_{H_2} \otimes ._M) = ._M \circ \beta \circ (\Delta_{H_2} \otimes id_M)$ and $\beta(h \otimes 1) = \varepsilon(h)1.$

Theorem

Let $\mathcal{K}$ be the sub bi-algebra of $\mathcal{H}$ generated by $\{\mathbf{1}, \partial_i^\star, (U_a^i)^{\pm 1}\}$. Then $\mathbb{C}_h\langle z\rangle$ is a left triple $(\mathbb{C}_h\langle z\rangle, \mathcal{K}, \mathcal{H})$-module algebra with respect to the action of $\mathbb{C}[[h]]$-linear maps $\triangleright_1 : \mathbb{C}_h\langle z\rangle \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ defined by $\triangleright_1(f \otimes g) = f \star g$ and $\triangleright_2 : \mathcal{K} \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ given by $\triangleright_2(F \otimes f) = F \triangleright_2 f$ for $F \in \mathcal{K},\ f \in \mathbb{C}_h\langle z\rangle$ where for $F = \partial_i^\star,\ \partial_i^\star \triangleright_2 f = \partial_i^\star \triangleright f$ and for $F = U_a^i,\ U_a^i \triangleright_2 f = U_a^i f$.

Definition

Let $Der_h\langle z\rangle$ be the $\mathbb{C}[[h]]$-vector space generated by $\partial_i^\star$'s. A $\mathbb{C}[[h]]$-linear map $\alpha : Der_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ is called a *$\star$-one form* on $\mathbb{C}_h\langle z\rangle$. The set of all $\star$-one forms is denoted by $\Omega_h^1\langle z\rangle$. It can be made into a right $\mathbb{C}_h\langle z\rangle$-module by the action $(\alpha f)(D) = \alpha D \star f$, for $f \in \mathbb{C}_h\langle z\rangle$ and $\alpha \in \Omega_h^1\langle z\rangle,\ D \in Der_h\langle z\rangle$.

Theorem

Let $\mathcal{K}$ be the sub bi-algebra of $\mathcal{H}$ generated by $\{\mathbf{1}, \partial_i^\star, (U_a^i)^{\pm 1}\}$. Then $\mathbb{C}_h\langle z\rangle$ is a left triple $(\mathbb{C}_h\langle z\rangle, \mathcal{K}, \mathcal{H})$-module algebra with respect to the action of $\mathbb{C}[[h]]$-linear maps $\triangleright_1 : \mathbb{C}_h\langle z\rangle \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ defined by $\triangleright_1(f \otimes g) = f \star g$ and $\triangleright_2 : \mathcal{K} \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ given by $\triangleright_2(F \otimes f) = F \triangleright_2 f$ for $F \in \mathcal{K},\ f \in \mathbb{C}_h\langle z\rangle$ where for $F = \partial_i^\star,\ \partial_i^\star \triangleright_2 f = \partial_i^\star \triangleright f$ and for $F = U_a^i,\ U_a^i \triangleright_2 f = U_a^i f$.

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Definition

Let $d_\star z_i : Der_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ be defined on generators by $d_\star z_i(\partial_j^\star) = \partial_j^\star \triangleright z_i = \delta_{ij}1,\ (i,j=1,...,N)$ where $d_\star z_i \in \Omega_h^1\langle z\rangle$. It induces a $\mathbb{C}[[h]]$-linear map $d_\star : \mathbb{C}_h\langle z\rangle \to \Omega_h^1\langle z\rangle$ defined by $d_\star(1)=0,\quad d_\star(z_i)=d_\star z_i$ and for $f\in\mathbb{C}_h\langle z\rangle,\quad d_\star(f)(\partial_j^\star)=\partial_j^\star \triangleright f$ where $d_\star(f)=d_\star f\in\Omega_h^1\langle z\rangle$.

Lemma

For $f\in\mathbb{C}_h\langle z\rangle$, we have $d_\star f=\sum_i (d_\star z_i)(\partial_i^\star \triangleright f)$ and any $\alpha\in\Omega_h^1\langle z\rangle$ can be written as $\alpha=\sum_i (d_\star z_i)\alpha_i$ where $\alpha_i=\alpha(\partial_i^\star)$.

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For $f \in \mathbb{C}_h\langle z\rangle$, we have $d_\star f = \sum_i (d_\star z_i)(\partial_i^\star \triangleright f)$ and any $\alpha \in \Omega_h^1\langle z\rangle$ can be written as $\alpha = \sum_i (d_\star z_i)\alpha_i$ where $\alpha_i = \alpha(\partial_i^\star)$.

Theorem

The map $d_\star$ satisfing the $\star$-deformed Leibnitz rule holds the following relations as the $\star$-differential map on $\mathbb{C}_h\langle z\rangle$:

$$d_\star(f\star g)(\partial_i^\star)=d_\star f(\partial_i^\star)\star U_2^{i+1,N}g+U_1^{1,i-1}U_1^{i+1,N}U_2^i f\star d_\star g(\partial_i^\star)$$

Definition

The pair $(\Omega_h^1\langle z\rangle,\ d_\star)$ together with properties of the pervious lemma and the present theorem is called the first order $\star$-differential calculus on $\mathbb{C}_h\langle z\rangle$.

Theorem

The map $d_\star$ satisfing the $\star$-deformed Leibnitz rule holds the following relations as the $\star$-differential map on $\mathbb{C}_h\langle z\rangle$:

$$d_\star(f\star g)(\partial_i^\star) = d_\star f(\partial_i^\star)\star U_2^{i+1,N} g + U_1^{1,i-1} U_1^{i+1,N} U_2^i f\star d_\star g(\partial_i^\star)$$

Definition

The pair $(\Omega_h^1\langle z\rangle,\ d_\star)$ together with properties of the pervious lemma and the present theorem is called the first order $\star$-differential calculus on $\mathbb{C}_h\langle z\rangle$.

Let $\varphi : \mathcal{H} \bigotimes \Omega^1_h\langle z\rangle \to \Omega^1_h\langle z\rangle$ be defined by

$$\varphi(h, (d_\star z_i)\alpha_i) = (d_\star z_i)(h \triangleright_{1,2} \alpha_i)$$

for $h \in \mathcal{H}, \quad d_\star z_i \in \Omega^1_h\langle z\rangle, \quad \alpha_i \in \mathbb{C}_h\langle z\rangle$ in the notation of the above lemma where $\triangleright_{1,2} = \triangleright_1$, if $h \in \mathbb{C}_h\langle z\rangle$ otherwise, $\triangleright_{1,2} = \triangleright_2$. Extend it $\mathbb{C}[[h]]$-linearly to the whole of $\mathcal{H} \bigotimes \Omega^1_h\langle z\rangle$.

Lemma

The map φ satisfies the following equality

$$\varphi(h_1 \star h_2, \sum_i (d_\star z_i)\alpha_i) = \varphi(h_1, \varphi(h_2, \sum_i (d_\star z_i)\alpha_i))$$

for $h_1, h_2 \in \mathcal{H}, \quad \sum_i (d_\star z_i)\alpha_i \in \Omega^1_h\langle z\rangle$.

Let $\varphi : \mathcal{H} \bigotimes \Omega_h^1\langle z\rangle \to \Omega_h^1\langle z\rangle$ be defined by

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for $h_1, h_2 \in \mathcal{H}$, $\sum_i (d_\star z_i)\alpha_i \in \Omega_h^1\langle z\rangle$.

Let $(\tilde{\mathcal{H}}, \star \circ \tau, \eta, \Delta, \varepsilon)$ be the opposite bi-algebra of $(\mathcal{H}, \star, \eta, \Delta, \varepsilon)$, i.e. only the $\star$-product of $\mathcal{H}$ is changed into $\star \circ \tau$ where $\tau : \mathcal{H} \bigotimes \mathcal{H} \to \mathcal{H} \bigotimes \mathcal{H}, \quad h_1 \otimes h_2 \mapsto h_2 \otimes h_1$ is twist operator.
Let $\tilde{\varphi} : \tilde{\mathcal{H}} \bigotimes \Omega^1_h\langle z\rangle \to \Omega^1_h\langle z\rangle$ be defined like φ.
Obviously, for $h_1, h_2 \in \mathcal{H}, \quad \alpha \in \Omega^1_h\langle z\rangle$,
$\tilde{\varphi}(h_1 \star h_2, \alpha) = \tilde{\varphi}(h_2, \tilde{\varphi}(h_1, \alpha)) = \varphi(h_2, \varphi(h_1, \alpha)) = \varphi(h_2 \star h_1, \alpha)$
So $\tilde{\varphi}$ defines a left action of $\tilde{\mathcal{H}}$ on $\Omega^1_h\langle z\rangle$.

Theorem

The compatibility of the left action $\tilde{\varphi}$ and the $\star$-differential map $d_\star$ is given by

$$d_\star \circ \triangleright_{1,2} = \tilde{\varphi} \circ (id \otimes d_\star)$$

This is called the twisted invariance of the $\star$-differential calculus $(\Omega^1_h\langle z\rangle, d_\star)$.

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This is called the twisted invariance of the $\star$-differential calculus $(\Omega^1_h\langle z\rangle, d_\star)$.

The pair $(\Omega^1_h\langle z\rangle,\ d_\star)$ is characterized by the following universality property:

Theorem

Let Γ be any right $\mathbb{C}_h\langle z\rangle$-module and $\delta : \mathbb{C}_h\langle z\rangle \to \Gamma$ be any $\mathbb{C}[[h]]$-linear map satisfying $\delta 1 = 0, \quad \delta f = \sum_i (\delta z_i)(\partial^\star_i \triangleright f)$. Then there is a unique right $\mathbb{C}_h\langle z\rangle$-module morphism $\psi : \Omega^1_h\langle z\rangle \to \Gamma$ such that $\delta = \psi \circ d_\star$.

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