Journal of Advanced Research in Pure Mathematics
Online ISSN: 1943-2380

Vol. 4, Issue. 2, 2012, pp. 97-111
doi: 10.5373/jarpm.931.051011

A $\star$-differential calculus on functionally quantized N-space

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Abstract. In this paper, we investigate the functional quantization interpretation of the coordinate algebra of the quantum N-space and prove the existence of a first order $\star$-differential calculus on it. The approach is based on the formalism of $\star$-product on a commutative algebra making it into a noncommutative space. The $\star$-derivatives are constructed. They satisfy $\star$-deformed Leibnitz rule and are contained in a bi-algebra. The concept of $\star$-one forms is introduced as the dual notion of the $\star$-derivatives. In this way a universal first order $\star$-differential calculus, consistent with the opposite bi-algebra is obtained.

Keywords: Functional quantization; $\star$-Derivatives; $\star$-Product; Triple module; Twisted invariance.

Mathematics Subject Classification 2010: 53D55, 32Gxx, 58B32.

1 Introduction

Differential calculus over algebras is remarkable in its numerous applications ranging from algebraic geometry to the physical theories. The standard and somewhat old approach to the study of smooth manifolds is by studying certain conditions on the algebra of functions on it. But, what is new is that the notion of observables which comes from physics and the concept of state of a physical system demand physical meanings of the elements of the function algebra. All information about classical physical systems are encoded in the corresponding algebra of observables. The formal approach towards this procedure is the study of differential calculus. It follows that differential calculus needed to describe physical systems is closely related to the corresponding function algebra.

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†Received: 10 May 2011, revised: 15 July 2011, accepted: 21 September 2011.

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The noncommutative version of the above theory is the geometric approach to noncommutative algebras and construction of spaces which are locally presented by noncommutative function algebras. This is the theory of noncommutative geometry [3] and have been studied by both mathematician and physicists [5, 10, 13]. One approach in this study is the quantization of space-time [1, 4, 11, 12, 14], using noncommutative geometry and quantum groups. This approach plays an important role in most problems of noncommutative field theory [5, 6, 7, 9].

After quantization of space-time, the space will no longer be set of points, however if we can have an interpretation of the quantum space as set of its points, then appropriate theories can be developed on the quantized spaces parallel to those on the classical manifolds. In [11] it is shown that functional quantization provides useful tool for this interpretation. This formalism represents the noncommutative structure of the commutative space and provides tools for the study of noncommutative field theory. According to [11], the functional quantization of a space is defined in the following way:

Let Ω_1 and Ω_2 be two sets and $x_0 \in \Omega_1$. Assume that A is a unital $\mathbb{C}$-algebra of complex valued functions on Ω_1 and $\mathcal{A}$ is an A-submodule of the unital A-algebra of an A-valued functions on Ω_2. Clearly, $\mathcal{A}$ can be considered as a $\mathbb{C}$-vector space of the algebra of complex valued functions on $\Omega_1 \times \Omega_2$. Assume that $\mathcal{A}$ admits an internal composition law $*$ such that $(\mathcal{A}, *)$ is an associative and not necessarily commutative unital A-algebra, with unit $\Omega_2 \to \mathbb{C}, \quad \omega \mapsto 1$. The restriction of an element $f \in \mathcal{A}$ to $\{x_0\} \times \Omega_2$ will be denoted by f_{x_0}. Let $B = \{f_{x_0} | \ f \in \mathcal{A}\}$ be a subalgebra of the algebra of complex valued functions on Ω_2. Under the above assumption $\mathcal{A}$ is called an (x_0, A)-*functional quantization* of B and $\Phi : \mathcal{A} \to B, \quad f \mapsto f(x_0)$, the *quantization map*.

The framework of this paper is based on a $\star$-product as the composition law of functionally quantized N-space. We consider the functional quantization of the free $\mathbb{C}$-algebra generated by coordinates $z_1, \dots, z_N$. It is seen that the composition law of this quantization comes from the quantum group symmetry of the quantum N-space of [8]. As we will see this $\star$-product plays an important role in developing concepts of the classical differential calculus to the level of functionally quantized N-space and hence provides useful tools for the study of its geometry. In this point of view, the first order $\star$-differential calculus on the functionally quantized space is constructed in the following steps.

The notion of $\star$-derivatives is defined in section 2. This is the noncommutative generalization of the notion of q^2-differentiations [8]. It is seen that these $\star$-derivatives satisfy the $\star$-deformed Leibnitz rule.

In section 3, we enlarge the functionally quantized N-space into a bi-algebra containing $\star$-derivatives. The notion of a triple module algebra is introduced. It is shown that this bi-algebra induces the structure of a triple module algebra on the functionally quantized N-space.

Finally in section 4, the notion of $\star$-one forms are introduced as the dual concept of $\star$-derivatives. These are made into a module over the functionally quantized N-space. This module structure is also enlarged, using the triple module formalism, to the level of a bi-algebra module. In this way a universal first order $\star$-differential calculus, twisted invariant with respect to the opposite bi-algebra is constructed.

2 Functionally quantized N-space and $\star$-derivatives

Let $\mathbb{C}\langle z_1, \dots, z_N\rangle$ be the free $\mathbb{C}$-algebra generated by N coordinates $z_1, \dots, z_N$. This is the associative, commutative, unital $\mathbb{C}$-algebra of formal power series in $z_1, \dots, z_N$. The quantum N-space is the quadratic algebra

$$\mathbb{C}_q^N = \frac{\mathbb{C}\langle z_1, \dots, z_N\rangle}{\mathcal{R}} \tag{2.1}$$

where $\mathcal{R}$ is the ideal generated by the relations $z_i z_j - q z_j z_i,\ i < j$ and $0 \neq q \in \mathbb{C}$, representing the quantum group structure of the quantum N-space [8]. Functions on this space are considered to be formal power series in $z_1, \dots, z_N$. The ideal $\mathcal{R}$ resembles the quantum group symmetry of $\mathbb{C}_q^N$. Set $\hat{z}_i = z_i$ (modulo $\mathcal{R}$), $i = 1, \dots, N$. Thereby in $\mathbb{C}_q^N$ the following relations are satisfied

$$\hat{z}_i \hat{z}_j = q \hat{z}_j \hat{z}_i, \quad i < j \tag{2.2}$$

In general for the monomials [2],

$$(\hat{z}_i)^{m_i} (\hat{z}_j)^{m_j} = q^{m_i m_j} (\hat{z}_j)^{m_j} (\hat{z}_i)^{m_i}, \quad i < j \tag{2.3}$$

$$(\hat{z}_1)^{l_1} \dots (\hat{z}_N)^{l_N} . (\hat{z}_1)^{k_1} \dots (\hat{z}_N)^{k_N} = q^{(-\sum_{i=1}^{N-1} k_i \sum_{j=i+1}^{N} l_j)} (\hat{z}_1)^{l_1+k_1} \dots (\hat{z}_N)^{l_N+k_N} \tag{2.4}$$

Set $q = e^h$ and let $\mathbb{C}[[h]]$ be the unital commutative algebra of formal power series in h with coefficients in $\mathbb{C}$. Addition and multiplication in $\mathbb{C}[[h]]$ are given formally by

$$a + b = \sum_{n=0}^{\infty} (a_n + b_n) h^n, \quad ab = \sum_{n=0}^{\infty} (\sum_{r=0}^{n} a_r b_{n-r}) h^n \tag{2.5}$$

for $a = \sum_{n=0}^{\infty} a_n h^n$ and $b = \sum_{n=0}^{\infty} b_n h^n$. In the h-adic topology, $\mathbb{C}\langle z_1, \dots, z_N\rangle \hat{\bigotimes} \mathbb{C}[[h]]$ is isomorphic to $\mathbb{C}\langle z_1, \dots, z_N\rangle[[h]]$, the algebra of formal power series in h with coefficients in $\mathbb{C}\langle z_1, \dots, z_N\rangle$ [8]. Having this in mind, $\mathbb{C}_q^N$ can be regarded as a $\mathbb{C}[[h]]$-algebra where the product of two functions on $\mathbb{C}_q^N$ will be a formal power series in $\hat{z}_1, \dots, \hat{z}_N$ with coefficients in $\mathbb{C}[[h]]$. We can have a functional quantization interpretation of this quantum N-space [11], i.e. the space $\mathbb{C}^N$ with coordinate functions $z_1, \dots, z_N$ has not been changed, but instead the algebra of functions on it is functionally quantized so that the product of functions no more remain commutative. This is done in the following way:
Let $\mathcal{C}$ be $\mathbb{C}[[h]]$-module of formal power series in $z_1, \dots, z_N$ with values in $\mathbb{C}[[h]]$. An element of $\mathcal{C}$ is of the form

$$f = f(z) = \sum_{i_1, \dots, i_N \geq 0} f_{i_1, \dots, i_N} (z_1)^{i_1} \dots (z_N)^{i_N} \tag{2.6}$$

with coefficients $f_{i_1, \dots, i_N} \in \mathbb{C}[[h]]$. Thanks to the Poincare-Birkhoff-Witt property, by considering the normal ordering of the coordinates $\hat{z}_1, \dots, \hat{z}_N$ in $\mathbb{C}_q^N$, the dimension of the subspace spanned by monomials of a fixed degree in $\hat{z}_1, \ldots, \hat{z}_N$ is the same as the dimension of the subspace spanned by monomials in the commutative variables $z_1, \ldots, z_N$ [14]. This together with (2.4) suggest a $\mathbb{C}$-vector space basis for $\mathbb{C}_q^N$ consisting of monomials of normal ordering, given by $\{(\hat{z}_1)^{i_1} \ldots (\hat{z}_N)^{i_N} | i_1, \ldots, i_N = 0, 1, 2, \ldots\}$. This leads to a $\mathbb{C}[[h]]$-module isomorphism $\mathbf{W} : \mathcal{C} \to \mathbb{C}_q^N$, defined on the generators by

$$\mathbf{W}((z_1)^{i_1} \ldots (z_N)^{i_N}) = (\hat{z}_1)^{i_1} \ldots (\hat{z}_N)^{i_N} \tag{2.7}$$

$\mathbf{W}$ can be extended to a $\mathbb{C}[[h]]$-algebra isomorphism by the following composition law on $\mathcal{C}$:

$$f \star g = \mathbf{W}^{-1}(\mathbf{W}(f).\mathbf{W}(g)) \tag{2.8}$$

for $f, g \in \mathcal{C}$. The relation (2.8) makes $\mathcal{C}$ into an associative, noncommutative unital $\mathbb{C}[[h]]$-algebra. In fact, $\mathcal{C}$ is the (0, $\mathbb{C}[[h]]$)-functional quantization of $\mathbb{C}\langle z_1, \ldots, z_N\rangle$ together with quantization map

$$\begin{aligned} &\Phi : \mathcal{C} \longrightarrow \mathbb{C}\langle z_1, \ldots, z_N\rangle \\ &\sum f_{i_1,\ldots,i_N}(z_1)^{i_1} \ldots (z_N)^{i_N} \to \sum f_{i_1,\ldots,i_N}(0)(z_1)^{i_1} \ldots (z_N)^{i_N} \end{aligned} \tag{2.9}$$

It is called the *functionally quantized N-space* and denoted by $\mathbb{C}_h\langle z\rangle$.
The image of $f \in \mathbb{C}_h\langle z\rangle$ in (2.6) under $\mathbf{W}$ is denoted by $\hat{f}$ which is

$$\hat{f} = \hat{f}(\hat{z}) = \sum_{i_1,\ldots,i_N \geq 0} f_{i_1,\ldots,i_N}(\hat{z}_1)^{i_1} \ldots (\hat{z}_N)^{i_N} \tag{2.10}$$

The commutative relation between elements of $\mathbb{C}_h\langle z\rangle$ is given by:

Lemma 2.1. The $\star$-product satisfies the following commutation relation

$$f \star g = q^{(\sum_{r=1}^{N} k_{N-r+1}(\sum_{s=1}^{N-r} l_s - \sum_{t=N-r+2}^{N} l_t))} g \star f \tag{2.11}$$

where $f = \sum f_{l_1,\ldots,l_N}(z_1)^{l_1} \ldots (z_N)^{l_N}$, $\quad g = \sum g_{k_1,\ldots,k_N}(z^1)^{k_1} \ldots (z^N)^{k_N}$.

Proof. It suffices to prove the formula for the monomials. From (2.4) and (2.8)

$$((z_1)^{l_1} \ldots (z_N)^{l_N}) \star ((z_1)^{k_1} \ldots (z_N)^{k_N}) = q^{(-\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} k_i l_j)} (z_1)^{l_1+k_1} \ldots (z_N)^{l_N+k_N}$$

and likewise,

$$((z_1)^{k_1} \ldots (z_N)^{k_N}) \star ((z_1)^{l_1} \ldots (z_N)^{l_N}) = q^{(-\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} l_i k_j)} (z_1)^{l_1+k_1} \ldots (z_N)^{l_N+k_N}$$

On the other hand,

$$-\sum_{i=1}^{N-1} k_i \sum_{j=i+1}^{N} l_j = \sum_{i=1}^{N} k_i (\sum_{j=1}^{i-1} l_j - \sum_{j=i+1}^{N} l_j) - \sum_{i=2}^{N} k_i \sum_{j=1}^{i-1} l_j = \sum_{i=1}^{N} k_i (\sum_{j=1}^{i-1} l_j - \sum_{j=i+1}^{N} l_j) -$$
$$\sum_{i=1}^{N-1} l_i \sum_{j=i+1}^{N} k_j = \sum_{r=1}^{N} k_{N-r+1} (\sum_{j=1}^{N-r} l_j - \sum_{j=N-r+2}^{N} l_j) - \sum_{i=1}^{N-1} l_i \sum_{j=i+1}^{N} k_j$$

This completes the proof. $\square$

The left multiplication Z_i of functions by coordinates and the scaling operators $U_a^i,\ U_a^{i,j}$ for $i,j=1,\ldots,N$, are defined by

$$(Z_if)(z)=z_if(z),\quad f\in\mathbb{C}_h\langle z\rangle \tag{2.12}$$

$$(U_a^if)(z)=f(q^az_i) \tag{2.13}$$

$$(U_a^{i,j}f)(z)=U_a^i\ldots U_a^jf(z),\quad i<j \tag{2.14}$$

For $i\geq j$ and $k<1$, $U_a^{i,j}$ and U_a^k are defined to be the identity operators.
The notions of q-calculus and q-differentiations on noncommutative spaces are introduced in [8]. We rewrite them on $\mathbb{C}_h\langle z\rangle$. For $a\in\mathbb{Z}$, the q-differentiation operator $D^i_{q^a}$ is given by

$$D^i_{q^a}f(z)=\mathbf{W}^{-1}(D^i_{q^a}\hat{f}(\hat{z}))=\mathbf{W}^{-1}(\frac{\hat{f}(\hat{z})-\hat{f}(q^a\hat{z}_i)}{(1-q^a)\hat{z}_i})=\frac{f(z)-f(q^az_i)}{(1-q^a)z_i} \tag{2.15}$$

for $f\in\mathbb{C}_h\langle z\rangle$, $\hat{f}=\mathbf{W}(f)$ and $\hat{f}(q^a\hat{z}_i)=\hat{f}(\hat{z}_1,\ldots,\hat{z}_{i-1},q^a\hat{z}_i,\hat{z}_{i+1},\ldots,\hat{z}_N)$. In the last equality we applied (2.7) and (2.10). In the limit case $q\to 1$, $D^i_{q^a}$ tends to the usual partial derivative ∂_i with respect to the i-th coordinate. For any integer l, let

$$[[l]]_{q^a}=\frac{1-q^{al}}{1-q^a}. \tag{2.16}$$

Clearly, $D^i_{q^a}(\hat{z}_i)^n=[[n]]_{q^a}(\hat{z}_i)^{n-1}$.

Lemma 2.2. The q-differentiations, the left multiplications and the scaling operators satisfy the following equations

$$(i)\ \ U_a^i(f\star g)=U_a^if\star U_a^ig,\quad U_a^{i,j}(f\star g)=(U_a^{i,j}f)\star(U_a^{i,j}g) \tag{2.17}$$

$$(ii)\ \ D^i_{q^a}Z_i-Z_iD^i_{q^a}=U_a^i \tag{2.18}$$

$$(iii)\ \ U_b^iD^i_{q^a}=q^{-a}D^i_{q^a}U_b^i,\quad U_b^jD^i_{q^a}=D^i_{q^a}U_b^j,\quad i\neq j \tag{2.19}$$

$$(iv)\ \ U_a^iZ_i=q^aZ_iU_a^i,\quad U_a^iZ_j=Z_jU_a^i,\quad i\neq j \tag{2.20}$$

$$(v)\ \ D^i_{q^a}D^j_{q^b}=q^{-1}D^j_{q^b}D^i_{q^a},\quad i<j \tag{2.21}$$

Proof. (i) is easily verified by using the definition (2.8) together with (2.13) and (2.14). For (ii) from (2.12) though (2.15) we have

$$(D^i_{q^a}Z_i-Z_iD^i_{q^a})(f(z))=(1-q^a)^{-1}(z_i)^{-1}((Z_if)(z)-(Z_if)(q^az_i)-Z_i(f(z)-f(q^az_i)))=$$
$$(1-q^a)^{-1}(f(z)-q^af(q^az^i)-f(z)+f(q^az^i))=(1-q^a)^{-1}(1-q^a)f(q^az^i)=f(q^az^i)=(U_a^if)(z).$$

To prove (iii) from (2.13) and (2.15),

$$(U_b^iD^i_{q^a}f)(z)=(D^i_{q^a}f)(q^bz_i)=(1-q^a)^{-1}(q^az_i)^{-1}(f(q^bz_i)-f(q^{a+b}z_i))=$$
$$q^{-a}(1-q^a)^{-1}(z_i)^{-1}((U_b^if)(z)-(U_b^if)(q^az_i))=q^{-a}(D^i_{q^a}U_b^if)(z).$$

The second assertion in (iii) is obvious. Also

$$(U_a^i Z_i f)(z) = (Z_i f)(q^a z_i) = q^a z_i f(q^a z_i) = q^a z_i (U_a^i f)(z) = q^a (Z_i U_a^i f)(z).$$

This proves the first assertion in (iv). Similarly, the second part is satisfied. For (v) from (2.15) we can write

$$D^i_{q^a}(D^j_{q^b} f(z)) = D^i_{q^a}((1-q^b)^{-1}(z_j)^{-1}(f(z) - f(q^b z_j))) = (1-q^b)^{-1}(D^i_{q^a}((z_j)^{-1} f(z)) -$$
$$D^i_{q^a}((z_j)^{-1} f(q^b z_j))) = (1-q^b)^{-1}(1-q^a)^{-1}(z_i)^{-1}(((z_j)^{-1} f(z) - (z_j)^{-1} f(q^a z_i)) - ((z_j)^{-1} f(q^b z_j) -$$
$$(z_j)^{-1} f(q^a z_i, q^b z_j))) = (1-q^a)^{-1}(1-q^b)^{-1}(z_j z_i)^{-1}(f(z) - f(q^a z_i) - f(q^b z_j) + f(q^a z_i, q^b z_j)) =$$
$$(1-q^a)^{-1}(1-q^b)^{-1} q^{-1}(z_j)^{-1}(z_i)^{-1}(f(z) - f(q^a z_i) - f(q^b z_j) + f(q^a z_i, q^b z_j)) =$$
$$q^{-1} D^j_{q^b}((1-q^a)^{-1}(z_i)^{-1}(f(z) - f(q^a z_i))) = q^{-1} D^j_{q^b}(D^i_{q^a} f(z)).$$

□

A first order differential calculus on the quantum N-space $\mathbb{C}_q^N$ is given in [8] by

$$\begin{aligned}
&\hat\partial_i \hat\partial_j = q^{-1} \hat\partial_j \hat\partial_i, \quad i<j \\
&\hat\partial_i \hat z_j = q \hat z_j \hat\partial_i, \quad i \neq j \\
&\hat\partial_i \hat z_i - q^2 \hat z_i \hat\partial_i = 1 + (q^2-1)\sum_{j>i} \hat z_j \hat\partial_j
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
&\hat z_i d\hat z_i = q^2 d\hat z_i \hat z_i \\
&\hat z_i d\hat z_j = q d\hat z_j \hat z_i + (q^2-1) d\hat z_i \hat z_j, \quad i<j \\
&\hat z_j d\hat z_i = q d\hat z_i \hat z_j, \quad i<j
\end{aligned} \tag{2.23}$$

From the above relations, the action of $\hat\partial_i$ on the powers of coordinates is

$$\begin{aligned}
&\hat\partial_i (\hat z_j)^{l_j} = q^{l_j} (\hat z_j)^{l_j} \hat\partial_i, \quad i \neq j \\
&\hat\partial_i (\hat z_i)^{l_i} = [[l_i]]_{q^2} (\hat z_i)^{l_i - 1} + q^{2l_i} (\hat z_i)^{l_i} \hat\partial_i + (q^2-1)[[l_i]]_{q^2} (\hat z_i)^{l_i-1} \sum_{j>i} (\hat z_j) \hat\partial_j
\end{aligned} \tag{2.24}$$

Definition 2.3. The q-partial derivative operators ∂_i, $i = 1, \dots N$ on $\mathbb{C}_h\langle z\rangle$ are defined by

$$\partial_i f = \mathbf{W}^{-1}(\hat\partial_i \hat f), \quad f \in \mathbb{C}_h\langle z\rangle \tag{2.25}$$

Remark 2.4. [2] Direct calculation and induction applied to (2.24) lead to the following Leibnitz rule for the monomials

$$\begin{aligned}
\partial_i (z_1)^{l_1} \dots (z_N)^{l_N} &= q^{(\sum_{k=1}^{i-1} l_k + \sum_{k=i+1}^{N} 2l_k)} D^i_{q^2} (z_1)^{l_1} \dots (z_N)^{l_N} + q^{(l_i + \sum_{k=1}^{i-1} l_k)} (z_1)^{l_1} \dots (z_N)^{l_N} \partial_i \\
&+ q^{\sum_{k=1}^{i-1} l_k} (q^2-1) D^i_{q^2} \sum_{j=0}^{N-i-1} q^{(\sum_{k=0}^{j-1} l_{N-k})} z_{N-j} . [(z_1)^{l_1} \dots (z_N)^{l_N} \partial_{N-j} \\
&\quad + (q^2-1) \sum_{r=0}^{j} q^{(\sum_{s=0}^{j-r-1} l_{N-r-s})} z_{N-r} D^{N-j}_{q^2} (z_1)^{l_1} \dots (z_N)^{l_N} \partial_{N-r}].
\end{aligned} \tag{2.26}$$

So for $f \in \mathbb{C}_h\langle z\rangle$,

$$\partial_i f = D^i_{q^2} U_1^{1,i-1} U_2^{i+1,N} f + U_1^{1,i-1} U_1^{i+1,N} U_2^i f \partial_i + (q^2-1) D^i_{q^2} U_1^{1,i-1} \sum_{j=0}^{N-i-1} z_{N-j}.$$

$$[U_1^{N-j+1,N} f \partial_{N-j} + (q^2-1) D^{N-j}_{q^2} \sum_{r=0}^{j} z_{N-r} U_1^{N-j+1,N} U_1^{N-j+1,N-r} f \partial_{N-r}]. \quad (2.27)$$

The problem with the operators ∂_i is that they are not consistent with the $\star$-product, in the sense that with respect to the $\star$-product, the Leibniz rule is not satisfied. As we will see the first summand in (2.27) is consistent with $\star$-product.

Definition 2.5. For $i = 1, \ldots, N$, the $\star$-derivative operators $\partial_i^\star$ on $\mathbb{C}_h\langle z\rangle$ are defined by

$$\partial_i^\star \triangleright f = D^i_{q^2} U_1^{1,i-1} U_2^{i+1,N} f, \quad (2.28)$$

In the special case where $f = z_j$ it is seen that $\partial_i^\star \triangleright z_j = \delta_{ij} 1$. This is the same covariant first order differential calculus over a quantum space with respect to the Hopf algebra $Gl_q(N)$ [2, 8].

Proposition 2.6. The $\star$-derivatives $\partial_i^\star$ satisfy the $\star$-deformed Leibniz rule, i.e. for $f, g \in \mathbb{C}_h\langle z\rangle$

$$\partial_i^\star \triangleright (f \star g) = (\partial_i^\star \triangleright f) \star (U_2^{i+1,N} g) + (U_1^{1,i-1} U_1^{i+1,N} U_2^i f) \star (\partial_i^\star \triangleright g) \quad (2.29)$$

Proof. It suffices to prove the relation for the monomials $f = (z_1)^{l_1} \ldots (z_N)^{l_N}$ and $g = (z_1)^{k_1} \ldots (z_N)^{k_N}$. From the definition (2.28) by using (2.17) and the pull back of (2.4) under $\mathbf{W}$,

$$\partial_i^\star \triangleright ((z_1)^{l_1} \ldots (z_N)^{l_N} \star (z_1)^{k_1} \ldots (z_N)^{k_N}) = D^i_{q^2} U_1^{1,i-1} U_2^{i+1,N} (q^{(-\sum_{r=1}^{N-1} k_r \sum_{s=r+1}^{N} l_s)}$$
$$(z_1)^{l_1+k_1} \ldots (z_N)^{l_N+k_N}) = q^{(-\sum_{r=1}^{N-1} k_r \sum_{s=r+1}^{N} l_s)} U_1^{1,i-1} U_2^{i+1,N} D^i_{q^2} ((z_1)^{l_1+k_1} \ldots (z_N)^{l_N+k_N})$$
$$= q^{(-\sum_{r=1}^{N-1} \sum_{s=r+1}^{N} k_r l_s)} U_1^{1,i-1} U_2^{i+1,N} [[l_i + k_i]]_{q^2} (z_1)^{l_1+k_1} \ldots (z_i)^{l_i+k_i-1} \ldots (z_N)^{l_N+k_N} \quad (2.30)$$

It is easy to see that $[[l_i + k_i]]_{q^2} = 1 + q^2 + \cdots + q^{2(l_i+k_i-1)} = [[l_i]]_{q^2} + q^{2l_i}[[k_i]]_{q^2}$. Set $l'_j = l_j,\ k'_j = k_j$ for $j \neq i$ and $l'_i = l_i - 1,\ k'_i = k_i - 1$ for $i > 1$. Then the identity (2.30) is equal to

$$q^{(-\sum_{r=1}^{N-1} k_r \sum_{s=r+1}^{N} l_s)} U_1^{1,i-1} U_2^{i+1,N} \{[[l_i]]_{q^2} q^{(\sum_{r=1}^{N-1} k_r \sum_{s=r+1}^{N} l'_s)} ((z_1)^{l_1} \ldots (z_i)^{l_i-1} \ldots (z_N)^{l_N}) \star$$
$$((z_1)^{k_1} \ldots (z_N)^{k_N}) + q^{2l_i} [[k_i]]_{q^2} q^{(\sum_{r=1}^{N-1} k'_r \sum_{s=r+1}^{N} l_s)} ((z_1)^{l_1} \ldots (z_N)^{l_N}) \star ((z_1)^{k_1} \ldots (z_i)^{k_i-1} \ldots$$
$$(z_N)^{k_N})\} = U_1^{1,i-1} U_2^{i+1,N} \{D^i_{q^2}((z_1)^{l_1} \ldots (z_N)^{l_N}) \star q^{(-\sum_{t=1}^{i-1} k_t)} ((z_1)^{k_1} \ldots (z_N)^{k_N}) + q^{(-\sum_{t=i+1}^{N} l_t)}.$$
$$q^{2l_i} ((z_1)^{l_1} \ldots (z_N)^{l_N}) \star D^i_{q^2}((z_1)^{k_1} \ldots (z_N)^{k_N})\} = (\partial_i^\star \triangleright ((z_1)^{l_1} \ldots (z_N)^{l_N})) \star U_1^{1,i-1} U_2^{i+1,N}$$
$$(U_1^{1,i-1})^{-1} ((z_1)^{k_1} \ldots (z_N)^{k_N}) + U_1^{1,i-1} U_2^{i+1,N} (U_1^{i+1,N})^{-1} U_2^i ((z_1)^{l_1} \ldots (z_N)^{l_N}) \star (\partial_i^\star \triangleright ((z_1)^{k_1}$$
$$\ldots (z_N)^{k_N})) = (\partial_i^\star \triangleright ((z_1)^{l_1} \ldots (z_N)^{l_N})) \star U_2^{i+1,N} ((z_1)^{k_1} \ldots (z_N)^{k_N}) + U_1^{1,i-1} U_1^{i+1,N} U_2^i ((z_1)^{l_1}$$
$$\ldots (z_N)^{l_N}) \star \partial_i^\star \triangleright ((z_1)^{k_1} \ldots (z_N)^{k_N}),$$

where in the last equality, we used the fact that $U_2^{i+1,N} (U_1^{i+1,N})^{-1} = U_1^{i+1,N}$. □

3 The formalism of triple module structure

In this section the quantized space $\mathbb{C}_h\langle z\rangle$ is enlarged into a bi-algebra. The bi-algebra equips $\mathbb{C}_h\langle z\rangle$ with a triple module structure, encoding its geometry. This will be shown in the next section. There we will see that the structure of $\star$-differential calculus on $\mathbb{C}_h\langle z\rangle$ is based on its triple module algebra. We start with the construction of the bi-algebra. Proposition (2.6) shows that the $\star$-derivatives are defined by their actions on functions. With the help of the $\star$-product of functions, we can define the $\star$-product between $\star$-derivatives, functions and scaling operators.

Definition 3.1. The $\star$-product between $\star$-derivatives $\partial_i^\star$, the scaling operators U_a^i and functions of $\mathbb{C}_h\langle z\rangle$ are defined by

$$(\partial_i^\star \star f) \triangleright g = \partial_i^\star \triangleright (f \star g) \tag{3.1}$$

$$(f \star \partial_i^\star) \triangleright g = f \star (\partial_i^\star \triangleright g) \tag{3.2}$$

$$(U_a^i \star \partial_i^\star) \triangleright f = U_a^i(\partial_i^\star \triangleright f), \quad (\partial_i^\star \star U_a^i) \triangleright f = \partial_i^\star \triangleright (U_a^i f) \tag{3.3}$$

$$(U_a^i \star U_b^j) \triangleright f = U_a^i U_b^j f, \quad (U_a^{i,j} \star U_b^{k,l}) \triangleright f = (U_a^{i,j} U_b^{k,l}) f \tag{3.4}$$

$$(\partial_i^\star \star \partial_j^\star) \triangleright f = \partial_i^\star \triangleright (\partial_j^\star \triangleright f) \tag{3.5}$$

$$(f \star U_a^i) \triangleright g = f \star U_a^i g, \quad (f \star U_a^{i,j}) \triangleright g = f \star U_a^{i,j} g \tag{3.6}$$

$$(U_a^i \star f) \triangleright g = U_a^i (f \star g), \quad (U_a^{i,j} \star f) \triangleright g = U_a^{i,j}(f \star g) \tag{3.7}$$

Remark 3.2. From the $\star$-deformed Leibnitz rule (2.29), the equation (3.1) can be written as

$$\partial_i^\star \star f = (\partial_i^\star \triangleright f) \star U_2^{i+1,N} + (U_1^{1,i-1} U_1^{i+1,N} U_2^i f) \star \partial_i^\star \tag{3.8}$$

Obviously, for $f = z_j$

$$\partial_i^\star \star z_j = q z_j \star \partial_i^\star, \quad j \neq i \tag{3.9}$$

and for $f = z_i$,

$$\partial_i^\star \star z_i - q^2 z_i \star \partial_i^\star = U_2^{i+1,N} \tag{3.10}$$

From (3.6) and (3.7) we have

$$U_a^i \star z_i = q^a z_i \star U_a^i, \quad U_a^i \star z_j = z_j \star U_a^i, \quad j \neq i \tag{3.11}$$

Relation (3.3) together with (2.19) give rise to the commutation rule between the scaling operators U_a^i's and the $\star$-derivatives

$$U_a^i \star \partial_i^\star = q^{-a} \partial_i^\star \star U_a^i \tag{3.12}$$

Relation (3.4) shows that the scaling operators U_a^i commute with each other

$$U_a^i \star U_b^j = U_b^j \star U_a^i \tag{3.13}$$

Relation (3.5) together with (2.21) lead to the commutativity relation for $\star$-derivatives

$$\partial_i^\star \star \partial_j^\star = q^{-1} \partial_j^\star \star \partial_i^\star, \quad i < j \tag{3.14}$$

Equation (3.8) defines a $\star$-commutator between $\star$-derivatives and functions as operators:

Definition 3.3. For $f \in \mathbb{C}_h\langle z\rangle$, the $\star$-commutator is defined by

$$[\partial_i^\star \overset{\star}{,} f] = \partial_i^\star \star f - (U_1^{1,i-1} U_1^{i+1,N} U_2^i f) \star \partial_i^\star \tag{3.15}$$

From (3.9) and (3.10) it is seen that for $f = z_j,\ j \neq i$,

$$[\partial_i^\star \overset{\star}{,} z_j] = \partial_i^\star \star z_j - q z_j \star \partial_i^\star = 0 \tag{3.16}$$

and for $f = z_i$,

$$[\partial_i^\star \overset{\star}{,} z_i] = U_2^{i+1,N} \tag{3.17}$$

Proposition 3.4. Let $(\mathcal{H}, \star)$ be the $\mathbb{C}[[h]]$-algebra with generators $\mathbf{1}$, the $\star$-derivatives $\partial_i^\star$'s, scaling operators U_a^i's, their inverse $(U_a^i)^{-1} = U_{-a}^i$ and functions on $\mathbb{C}_h\langle z\rangle$, with the $\star$-product structure of (3.1) though (3.7). Let $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H},\ \varepsilon : \mathcal{H} \to \mathbb{C}$ be defined on the generators by

$$\begin{aligned}
&\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}, \quad \Delta(\partial_\rho^\star) = \partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star, \\
&\Delta((U_a^\rho)^{\pm 1}) = (U_a^\rho)^{\pm 1} \otimes (U_a^\rho)^{\pm 1}, \quad \Delta(z^\rho) = z^\rho \otimes \mathbf{1} + \mathbf{1} \otimes z^\rho. \qquad (3.18) \\
&\varepsilon(\mathbf{1}) = \varepsilon(\partial_\rho^\star) = \varepsilon(z^\rho) = 0, \quad \varepsilon((U_a^\rho)^{\pm 1}) = 1. \qquad (3.19)
\end{aligned}$$

Then $(\mathcal{H}, \star, \eta, \Delta, \varepsilon)$ is a bi-algebra, with $\eta : \mathbb{C} \to \mathcal{H}$ the unit map $\eta(1) = \mathbf{1}$.

Proof. We extend the coproduct Δ (counit ε) on the $\star$-product of generators to be the $\star$-product of their coproducts (counits) using the following $\star$-product convention on $\mathcal{H} \otimes \mathcal{H}$,

$$(F_1 \otimes G_1) \star (F_2 \otimes G_2) = (F_1 \star F_2) \otimes (G_1 \star G_2)$$

where $F_i, G_i \in \mathcal{H}$ are generators. This makes Δ , ε are algebra homomorphisms. It is seen that they are both consistent with the $\star$-product, i.e. with the commutation relations (3.12) though (3.14). For $\mu < \nu$,

$$\Delta(\partial_\mu^\star \star \partial_\nu^\star) = \Delta(\partial_\mu^\star) \star \Delta(\partial_\nu^\star) = (\partial_(3.6)^\star \otimes U_2^{\mu+1,N} + U_1^{1,\mu-1} U_1^{\mu+1,N} U_2^\mu \otimes \partial_\mu^\star) \star (\partial_\nu^\star \otimes U_2^{\nu+1,N} + U_1^{1,\nu-1} U_1^{\nu+1,N} U_2^\nu \otimes \partial_\nu^\star) = \partial_\mu^\star \star \partial_\nu^\star \otimes U_2^{\mu+1,N} U_2^{\nu+1,N} + U_1^{1,\mu-1} U_1^{\mu+1,N} U_2^\mu U_1^{1,\nu-1} U_1^{\nu+1,N} U_2^\nu \otimes \partial_\mu^\star \star \partial_\nu^\star.$$

On the other hand,

$$\Delta(q^{-1}\partial_\nu^\star \star \partial_\mu^\star) = q^{-1}\Delta(\partial_\nu^\star) \star \Delta(\partial_\mu^\star) = q^{-1}(\partial_\nu^\star \otimes U_2^{\nu+1,N} + U_1^{1,\nu-1} U_1^{\nu+1,N} U_2^\nu \otimes \partial_\nu^\star) \star (\partial_\mu^\star \otimes U_2^{\mu+1,N} + U_1^{1,\mu-1} U_1^{\mu+1,N} U_2^\mu \otimes \partial_\mu^\star) = q^{-1}(\partial_\nu^\star \star \partial_\mu^\star \otimes U_2^{\nu+1,N} U_2^{\mu+1,N} + U_1^{1,\nu-1} U_1^{\nu+1,N} U_2^\nu U_1^{1,\mu-1} U_1^{\mu+1,N} U_2^\mu \otimes \partial_\nu^\star \star \partial_\mu^\star).$$

So we see the equality

$$\Delta(\partial_\mu^\star \star \partial_\nu^\star) = \Delta(\partial_\mu^\star) \star \Delta(\partial_\nu^\star) = q^{-1}\Delta(\partial_\nu^\star) \star \Delta(\partial_\mu^\star).$$

In the same way from (3.12) and (3.13)

$$\Delta(U_a^\rho \star \partial_\rho^\star) = \Delta(U_a^\rho) \star \Delta(\partial_\rho^\star) = (U_a^\rho \otimes U_a^\rho) \star (\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = U_a^\rho \star \partial_\rho^\star \otimes U_a^\rho U_2^{\rho+1,N} + U_a^\rho U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes U_a^\rho \star \partial_\rho^\star.$$

Which is equal to

$$q^{-a}\Delta(\partial_\rho^\star \star U_a^\rho) = q^{-a}\Delta(\partial_\rho^\star) \star \Delta(U_a^\rho) = q^{-a}(\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) \star (U_a^\rho \otimes U_a^\rho) = q^{-a}\partial_\rho^\star \star U_a^\rho \otimes U_a^\rho U_a^{\rho+1,N} + U_a^\rho U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes q^{-a}\partial_\rho^\star \star U_a^\rho.$$

Finally, from (3.13)

$$\Delta(U_a^\rho \star U_b^\mu) = \Delta(U_a^\rho) \star \Delta(U_b^\mu) = (U_a^\rho \otimes U_a^\rho) \star (U_b^\mu \otimes U_b^\mu) = U_a^\rho \star U_b^\mu \otimes U_a^\rho \star U_b^\mu = U_b^\mu \star U_a^\rho \otimes U_b^\mu \star U_a^\rho = \Delta(U_b^\mu \star U_a^\rho) = \Delta(U_b^\mu) \star \Delta(U_a^\rho).$$

We can also see that for $f \in \mathbb{C}_h\langle z\rangle$,

$$\Delta(f) = f \otimes 1 + 1 \otimes f,\ \varepsilon(f) = 0. \tag{3.20}$$

And so

$$\Delta(f \star g) = \Delta(f) \star \Delta(g) = (f \otimes \mathbf{1} + \mathbf{1} \otimes f) \star (g \otimes \mathbf{1} + \mathbf{1} \otimes g) = (f \star g) \otimes (\mathbf{1} \star \mathbf{1}) + (\mathbf{1} \star \mathbf{1}) \otimes (f \star g) = (f \star g) \otimes \mathbf{1} + \mathbf{1} \otimes (f \star g),$$

where $\mathbf{1} \star \mathbf{1}$ is equal to $\mathbf{1}$. Moreover,

$$\Delta(f \star \partial_\rho^\star) = \Delta(f) \star \Delta(\partial_\rho^\star) = (f \otimes 1 + 1 \otimes f) \star (\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = f \star \partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes f \star \partial_\rho^\star.$$

And we have

$$\Delta(f \star U_a^\rho) = \Delta(f) \star \Delta(U_a^\rho) = (f \otimes 1 + 1 \otimes f) \star (U_a^\rho \otimes U_a^\rho) = f \star U_a^\rho \otimes U_a^\rho + U_a^\rho \otimes f \star U_a^\rho.$$

$\Delta(\partial_\rho^\star \star f)$ and $\Delta(U_a^\rho \star f)$ are calculated in the same way. To show the coassociativity of Δ, we apply $(\Delta \otimes id) \circ \Delta$ to the generators

$$((\Delta \otimes id) \circ \Delta)(\partial_\rho^\star) = (\Delta \otimes id)(\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = (\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star = \partial_\rho^\star \otimes U_2^{\rho+1,N} \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes (\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = (id \otimes \Delta)(\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = ((id \otimes \Delta) \circ \Delta)(\partial_\rho^\star).$$

The same is true for the grouplike elements $(U_a^\rho)^{\pm 1}$ and the primitive elements f. Also for the counit ε

$$((\varepsilon \otimes id) \circ \Delta)(\partial_\rho^\star) = (\varepsilon \otimes id)(\partial_\rho^\star \otimes U_2^{\rho+1,N} + U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho \otimes \partial_\rho^\star) = \varepsilon(\partial_\rho^\star) U_2^{\rho+1,N} + \varepsilon(U_1^{1,\rho-1})\varepsilon(U_1^{\rho+1,N})\varepsilon(U_2^\rho)\partial_\rho^\star = \partial_\rho^\star.$$

It is seen that $(id \otimes \varepsilon) \circ \Delta$ gives the same result. For other generators it is easily seen to be true as well. $\square$

Definition 3.5. Let H be a bi-algebra and $H_1,\ H_2$ its sub bi-algebras. A *left triple* (H_1, H_2, H)*-module algebra* is a unital $\mathbb{C}$-algebra M such that

- M is a left module algebra with respect to H_1, i.e. there exists a $\mathbb{C}$-linear map $\alpha : H_1 \bigotimes M \to M$ such that $\alpha(a_1 \otimes \alpha(b_1 \otimes m)) = \alpha(a_1.b_1 \otimes m),\ \ a_1, b_1 \in H_1,\ m \in M$ and $\alpha(1 \otimes m) = m$.

- M is a left module bi-algebra with respect to H_2 in the sense that there exists a $\mathbb{C}$-linear map $\beta : H_2 \bigotimes M \to M$ such that $\beta(a_2 \otimes \beta(b_2 \otimes m)) = \beta(a_2.b_2 \otimes m),\ \ a_2, b_2 \in H_2,\ m \in M,\ \beta(1 \otimes m) = m,\ \beta(h \otimes a.b) = \sum \beta(h_{(1)} \otimes a)\beta(h_{(2)} \otimes b)$ and $\beta(h \otimes 1) = \varepsilon(h)1$ where we write the Sweedler notation for $\Delta,\ \Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

Proposition 3.6. Let $\mathcal{K}$ be the sub bi-algebra of $\mathcal{H}$ generated by $\{\mathbf{1}, \partial_i^\star, (U_a^i)^{\pm 1},\ i = 1, \ldots, N\}$. Then $\mathbb{C}_h\langle z\rangle$ is a left triple $(\mathbb{C}_h\langle z\rangle, \mathcal{K}, \mathcal{H})$-module algebra.

Proof. It follows from proposition (3.4) and relation (3.20) that $\mathbb{C}_h\langle z\rangle$ is a sub bi-algebra of $\mathcal{H}$. Let the $\mathbb{C}[[h]]$-linear map $\triangleright_1 : \mathbb{C}_h\langle z\rangle \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ be defined by

$$\triangleright_1(f \otimes g) = f \star g, \quad f, g \in \mathbb{C}_h\langle z\rangle \tag{3.21}$$

Obviously,

$$\triangleright_1 \circ (id_{\mathbb{C}_h\langle z\rangle} \otimes \triangleright_1) = \triangleright_1 \circ (\star \otimes id_{\mathbb{C}_h\langle z\rangle}).$$

and $\triangleright_1(1 \otimes f) = 1 \star f = f$. This makes $\mathbb{C}_h\langle z\rangle$ into a left module algebra.
Define the $\mathbb{C}[[h]]$-linear map $\triangleright_2 : \mathcal{K} \otimes \mathbb{C}_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ by

$$\triangleright_2(F \otimes f) = F \triangleright_2 f \tag{3.22}$$

for $F \in \mathcal{K},\ f \in \mathbb{C}_h\langle z\rangle$ where for $F = \partial_i^\star,\ \partial_i^\star \triangleright_2 f = \partial_i^\star \triangleright f$ and for $F = U_a^i,\ U_a^i \triangleright_2 f = U_a^i f$. Relations (3.3) through (3.5) show that

$$\triangleright_2 \circ (id_{\mathcal{K}} \otimes \triangleright_2) = \triangleright_2 \circ (\star \otimes id_{\mathbb{C}_h\langle z\rangle}).$$

Relations (2.17) and (2.29) can be rewritten as

$$\partial_\rho^\star \triangleright (f \star g) = \star \circ (\Delta(\partial_\rho^\star) \triangleright f \otimes g) = \star \circ ((\partial_\rho^\star \triangleright f) \otimes (U_2^{\rho+1,N} g) + (U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho f) \otimes (\partial_\rho^\star \triangleright g))$$
$$= (\partial_\rho^\star \triangleright f) \star (U_2^{\rho+1,N} g) + (U_1^{1,\rho-1} U_1^{\rho+1,N} U_2^\rho f) \star (\partial_\rho^\star \triangleright g).$$
$$(U_a^\rho)^{\pm 1} \triangleright (f \star g) = \star \circ (\Delta((U_a^\rho)^{\pm 1}) \triangleright f \otimes g) = \star \circ ((U_a^\rho)^{\pm 1} f \otimes (U_a^\rho)^{\pm 1} g) = (U_a^\rho)^{\pm 1} f \star (U_a^\rho)^{\pm 1} g$$
$$= (U_a^\rho)^{\pm 1} (f \star g).$$

Consequently,

$$\triangleright_2(F \otimes f \star g) = \sum \triangleright_2(F_{(1)} \otimes f) \star \triangleright_2(F_{(2)} \otimes g), \quad \triangleright_2(1 \otimes f) = f,$$

where $\Delta(F) = F_{(1)} \otimes F_{(2)}$. Also

$$\triangleright_2(\partial_\rho^\star \otimes 1) = \partial_\rho^\star \triangleright_2 1 = 0 = \varepsilon(\partial_\rho^\star)1$$
$$\triangleright_2((U_a^\rho)^{\pm 1} \otimes 1) = (U_a^\rho)^{\pm 1} \triangleright_2 1 = 1 = \varepsilon((U_a^\rho)^{\pm 1})1.$$

This shows that $\mathbb{C}_h\langle z\rangle$ is a left module bi-algebra with respect to $\mathcal{K}$. $\square$

4 Twisted invariant structure and universal first order $\star$-differential calculus

So far the triple module structure of $\mathbb{C}_h\langle z\rangle$ is established. Now we are equipped enough to study its geometry. In this section the notion of $\star$-one forms are introduced as the dual concept of $\star$-derivatives. Following this, the $\star$-differential calculus is constructed. The triple module formalism on $\mathbb{C}_h\langle z\rangle$ induces an $\mathcal{H}$-module structure on the space of $\star$-one forms which is invariant with respect to the opposite bi-algebra, so called twisted invariant. The universality property of the $\star$-differential calculus is a consequence of these structures.

Definition 4.1. Let $Der_h\langle z\rangle$ be the $\mathbb{C}[[h]]$-vector space generated by $\{\partial_i^\star,\ \ i=1,\dots,N\}$. A $\mathbb{C}[[h]]$-linear map $\alpha : Der_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ is called a *$\star$-one form* on $\mathbb{C}_h\langle z\rangle$. The set of all $\star$-one forms is denoted by $\Omega_h^1\langle z\rangle$.

$\Omega_h^1\langle z\rangle$ can be made into a right $\mathbb{C}_h\langle z\rangle$-module by the following action

$$(\alpha f)(D) = \alpha D \star f, \tag{4.1}$$

for $f \in \mathbb{C}_h\langle z\rangle$ and $\alpha \in \Omega_h^1\langle z\rangle,\ D \in Der_h\langle z\rangle$.

Definition 4.2. For $i = 1,\dots,N$, let $d_\star z_i : Der_h\langle z\rangle \to \mathbb{C}_h\langle z\rangle$ be defined on generators by

$$d_\star z_i(\partial_j^\star) = \partial_j^\star \triangleright z_i = \delta_{ij}1. \tag{4.2}$$

So $d_\star z_i \in \Omega_h^1\langle z\rangle$. It induces a $\mathbb{C}[[h]]$-linear map $d_\star : \mathbb{C}_h\langle z\rangle \to \Omega_h^1\langle z\rangle$ defined by $d_\star(1) = 0,\ \ d_\star(z_i) = d_\star z_i$ and for $f \in \mathbb{C}_h\langle z\rangle$,

$$d_\star(f)(\partial_j^\star) = \partial_j^\star \triangleright f \tag{4.3}$$

Obviously, $d_\star(f) = d_\star f \in \Omega_h^1\langle z\rangle$.

Lemma 4.3. For $f \in \mathbb{C}_h\langle z\rangle$, we have

$$d_\star f = \sum_i (d_\star z_i)(\partial_i^\star \triangleright f) \tag{4.4}$$

Proof. Since both sides are $\mathbb{C}[[h]]$-linear maps, it suffices to check the equality for the basis elements $\partial_i^\star$. From the second equality in (4.1) we have

$$(\sum_i (d_\star z_i)(\partial_i^\star \triangleright f))(\partial_j^\star) = \sum_i (d_\star z_i)(\partial_j^\star) \star (\partial_i^\star \triangleright f) = \partial_j^\star \triangleright f = d_\star f(\partial_j^\star).$$

□

Proposition 4.4. The map $d_\star$ satisfies the $\star$-deformed Leibniz rule (2.29) in the sense that it satisfies the following relations

$$d_\star(f \star g)(\partial_i^\star) = d_\star f(\partial_i^\star) \star U_2^{i+1,N} g + U_1^{1,i-1} U_1^{i+1,N} U_2^i f \star d_\star g(\partial_i^\star) \tag{4.5}$$

It is called the $\star$-differential map on $\mathbb{C}_h\langle z\rangle$.

Proof. The proof is by direct calculation applying relations (4.3), (2.29) and the associativity property of $\star$-product. □

Remark 4.5. Any $\alpha \in \Omega_h^1\langle z\rangle$ can be written as $\alpha = \sum_i (d_\star z_i)\alpha_i$ where $\alpha_i = \alpha(\partial_i^\star)$. This is true since on the basis elements $\partial_j^\star$ we have

$$(\sum_i (d_\star z_i)\alpha_i)(\partial_j^\star) = \sum_i (d_\star z_i)(\partial_j^\star)\alpha_i = \alpha_j = \alpha(\partial_j^\star)$$

where in the first equality we used (4.1) and in the last equality we used the fact that α is $\mathbb{C}[[h]]$-linear.

Definition 4.6. The pair $(\Omega_h^1\langle z\rangle,\ d_\star)$ together with properties of definition (4.3), proposition (4.4) and remark (4.5) is called the first order $\star$-differential calculus on $\mathbb{C}_h\langle z\rangle$.

In what follows we see that there exists a left action of the opposite bi-algebra of $\mathcal{H}$ on $\Omega_h^1\langle z\rangle$ which is compatible with the $\star$-differential map $d_\star$.
Let $\varphi : \mathcal{H} \bigotimes \Omega_h^1\langle z\rangle \to \Omega_h^1\langle z\rangle$ be defined by

$$\varphi(h, (d_\star z_i)\alpha_i) = (d_\star z_i)(h \triangleright_{1,2} \alpha_i) \tag{4.6}$$

for $h \in \mathcal{H},\ \ d_\star z_i \in \Omega_h^1\langle z\rangle,\ \ \alpha_i \in \mathbb{C}_h\langle z\rangle$ in the notation of the remark (4.5) where $\triangleright_{1,2} = \triangleright_1$, if $h \in \mathbb{C}_h\langle z\rangle$ otherwise, $\triangleright_{1,2} = \triangleright_2$. Extend it $\mathbb{C}[[h]]$-linearly to the whole of $\mathcal{H} \bigotimes \Omega_h^1\langle z\rangle$.

Lemma 4.7. The map φ satisfies the following equality

$$\varphi(h_1 \star h_2, \sum_i (d_\star z_i)\alpha_i) = \varphi(h_1, \varphi(h_2, \sum_i (d_\star z_i)\alpha_i)) \tag{4.7}$$

for $h_1, h_2 \in \mathcal{H},\quad \sum_i (d_\star z_i)\alpha_i \in \Omega_h^1\langle z\rangle$.

Proof. From the associativity of the $\star$-product for $\triangleright = \triangleright_1$ and relations (3.1) though (3.7) for $\triangleright = \triangleright_2$,

$$\varphi(h_1 \star h_2, (d_\star z_i)\alpha_i) = (d_\star z_i)((h_1 \star h_2) \triangleright \alpha_i) = (d_\star z_i)(h_1 \triangleright (h_2 \triangleright \alpha_i)) = \varphi(h_1, (d_\star z_i)(h_2 \triangleright \alpha_i)) = \varphi(h_1, \varphi(h_2, (d_\star z_i)\alpha_i))$$

□

Let $(\tilde{\mathcal{H}}, \star \circ \tau, \eta, \Delta, \varepsilon)$ be the opposite bi-algebra of $(\mathcal{H}, \star, \eta, \Delta, \varepsilon)$, i.e. only the $\star$-product of $\mathcal{H}$ is changed into $\star \circ \tau$ where $\tau : \mathcal{H} \bigotimes \mathcal{H} \to \mathcal{H} \bigotimes \mathcal{H},\quad h_1 \otimes h_2 \mapsto h_2 \otimes h_1$ is twist operator. Let $\tilde{\varphi} : \tilde{\mathcal{H}} \bigotimes \Omega_h^1\langle z\rangle \to \Omega_h^1\langle z\rangle$ be defined as in (4.6). Obviously for $h_1, h_2 \in \mathcal{H},\quad \alpha \in \Omega_h^1\langle z\rangle$,

$$\tilde{\varphi}(h_1 \star h_2, \alpha) = \tilde{\varphi}(h_2, \tilde{\varphi}(h_1, \alpha)) = \varphi(h_2, \varphi(h_1, \alpha)) = \varphi(h_2 \star h_1, \alpha) \tag{4.8}$$

So $\tilde{\varphi}$ defines a left action of $\tilde{\mathcal{H}}$ on $\Omega_h^1\langle z\rangle$.

Proposition 4.8. In the above notations and conventions, the compatibility of the left action $\tilde{\varphi}$ and the $\star$-differential map $d_\star$ is given by the commutativity of the following diagram.

$$\begin{array}{ccc} \mathcal{H} \bigotimes \mathbb{C}_h\langle z\rangle & \xrightarrow{\triangleright_{1,2}} & \mathbb{C}\langle z\rangle \\ {\scriptstyle id\otimes d_\star}\Big\downarrow & & \Big\downarrow{\scriptstyle d_\star} \\ \tilde{\mathcal{H}} \bigotimes \Omega^1_h\langle z\rangle & \xrightarrow{\tilde{\varphi}} & \Omega^1_h\langle z\rangle \end{array}$$

This is called the twisted invariance of the $\star$-differential calculus $(\Omega^1_h\langle z\rangle, d_\star)$.

Proof. For $h \in \mathcal{H}, \quad f \in \mathbb{C}_h\langle z\rangle$ we have

$$\tilde{\varphi}(h, d_\star f) = \tilde{\varphi}(h, \sum_i (d_\star z_i)(\partial_i^\star \triangleright f)) = \sum_i \tilde{\varphi}(h, \varphi(\partial_i^\star, (d_\star z_i)f)) = \sum_i \varphi(\partial_i^\star \star h, (d_\star z_i)f) =$$
$$\sum_i (d_\star z_i)(\partial_i^\star \star h) \triangleright f = \sum_i (d_\star z_i)(\partial_i^\star \triangleright (h \triangleright_{1,2} f) = d_\star(h \triangleright_{1,2} f).$$

□

The pair $(\Omega^1_h\langle z\rangle,\ d_\star)$ is characterized by the following universality property:

Proposition 4.9. Let Γ be any right $\mathbb{C}_h\langle z\rangle$-module and $\delta : \mathbb{C}_h\langle z\rangle \to \Gamma$ be any $\mathbb{C}[[h]]$-linear map satisfying $\delta 1 = 0, \quad \delta f = \sum_i (\delta z_i)(\partial_i^\star \triangleright f)$. Then there is a unique right $\mathbb{C}_h\langle z\rangle$-module morphism $\psi : \Omega^1_h\langle z\rangle \to \Gamma$ such that $\delta = \psi \circ d_\star$.

Proof. Define ψ on the generators $d_\star z_i$ by $\psi(d_\star z_i) = \delta z_i$ and extend it to a right $\mathbb{C}_h\langle z\rangle$-module morphism on $\Omega^1_h\langle z\rangle$ by

$$\psi(\sum_i (d_\star z_i)\alpha_i) = \sum_i (\delta z_i)\alpha_i \tag{4.9}$$

for $\sum_i (d_\star z_i)\alpha_i \in \Omega^1_h\langle z\rangle$. Clearly, $\psi \circ d_\star = \delta$. Furthermore if ψ' is another right $\mathbb{C}_h\langle z\rangle$-module homomorphism with $\delta = \psi' \circ d_\star$ then we must have

$$\psi'(\sum_i (d_\star z_i)\alpha_i) = \sum_i \psi'(d_\star z_i)\alpha_i = \sum_i (\delta z_i)\alpha_i = \psi(\sum_i (d_\star z_i)\alpha_i).$$

So ψ is unique. □

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