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The Space of Integrable Dirac Structures on Hilbert C*-Modules

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Abstract

In this paper we interpret the integrability of the Dirac structures on some Hilbert C*-modules in terms of an automorphism group. This is the group of orthogonal transformations on the Hilbert C*-module of sections of a Hermitian vector bundle over an smooth manifold M. Some topological properties of the group of integrable Dirac structures are studied. In some special cases it is shown that the integrability condition corresponds to the solutions of a partial differential equation. This is explained as a necessary and sufficient condition.

Key words: C^*-algebra, Dirac structure, isotropic, Hermitian vector bundle, Hilbert C^*-module

AMS subject class: 46B20, 46C05, 53B35

1 Introduction

The idea of a Poisson bracket on the algebra of smooth functions on a smooth manifold M goes back to Dirac [4]. The underlying structure of any Hamiltonian system is a Poisson algebra. Courant and Weinstein [2] presented an approach to unify the geometry of Hamiltonian vector fields and the Poisson brackets (unification of Poisson and symplectic geometry). In both of these geometries, the Poisson algebra is $C^\infty(M)$ and the bracket is given by a specific bivector field on M [4].

As a generalization of Poisson and presymplectic structures, the theory of Dirac structures on vector spaces and their extension to manifolds was considered by Courant and Weinstein [1,2]. These are smooth subbundles of the direct sum bundle $TM \oplus T^*M$ of the tangent and cotangent bundles, maximally isotropic under the pairing

$$< (X, \omega), (Y, \mu) >_+ =: \frac{\omega(Y) + \mu(X)}{2}$$

on $TM \oplus T^*M$.

Dorfman [5] developed the algebraic version of Dirac structures. The generalization of Dirac structures on real and complex Hilbert spaces and on Hermitian modules are considered in [7,8].

Our motivation in this paper has been the following consideration:

The integrability of Dirac structures on manifolds was introduced by Courant [1]. It is important in that it leads to a Poisson algebra of functions, making it possible to construct the classical mechanics on the manifold [3,4,5].

The object of this paper is the interpretation of integrable Dirac structures on pre-Hilbert C*-modules in such a way that we can specify the moduli space of the group of integrable Dirac structures on some Hilbert C*-modules.. The paper is organized in the following manner.

First we give some preliminaries on the basic concepts of Dirac structures on modules and on TM; the tangent bundle of the smooth manifold M and then introduce the notion of integrable Dirac structures on modules. After that we show in details how a Dirac structure on TM can be constructed out of an orthogonal transformation of a Hilbert C*-module; the module of sections of a Hermitian vector bundle on M. This enables us to define the integrability of Dirac structures in terms of the orthogonal transformations and go through their topological properties. A necessary and sufficient condition for the integrability of a Dirac structure is obtained as a solution to some certain partial differential equation.

2 Dirac Structures on pre-Hilbert C*-modules

The concepts in this section are based on the references [6,8].

Let $\mathcal{A}$ be a C*-algebra, H a right $\mathcal{A}$-module. The action of an element $a \in \mathcal{A}$ on H is denoted by $x.a$ for $x \in H$. H together with a sesquilinear form $<,>: H \times H \to \mathcal{A}$ with the following properties

i)$< x, x > \geq 0; \forall x \in H$

ii)$< x, x >= 0$, implies $x = 0$.

iii)$< x, y >^* =< y, x >; \forall x, y \in H$.

iv)$< x, y.a >=< x, y > a; \forall x, y \in H; \forall a \in \mathcal{A}$.

is called a *pre-Hilbert module.* For $x \in H$ let $\| x \|_H =: \| < x, x > \|^{1/2}$. If the normed space $(H, \| - \|)$ is complete, then H is called a *Hilbert C*-module.*

In this paper all the Hilbert C*-modules have the property that for each nonzero $x \in H$, $2x \neq 0$.

Example 2.1. Let M be a smooth compact n-manifold and $\pi : E \to M$ be a Hermitian vector bundle over M. Let $\mathcal{A}$ be the C*-algebra of continuous functions on M. Let H be the $\mathcal{A}$-module of sections of this vector bundle. Then H becomes a pre-Hilbert $\mathcal{A}$-module. In particular when $\pi : TM \to M$ is the tangent bundle, the Hermitian inner product enables us to identify this bundle with its dual T^*M and $\Gamma(TM)$; the $\mathcal{A}$-module of vector fields on M is identified with its dual $\Gamma(T^*M)$; the $\mathcal{A}$-module of first order differential forms on M.

Definition 2.2. Let H be a pre-Hilbert C*-module. Let $\tau : H \times H \to H \times H$ be the flip operator defined by $\tau(x, y) = (y, x)$ for $x, y \in H$. A submodule $L \subset H \times H$ is called a *Dirac structure on* H if L and $\tau(L)$ are ortho-complementary.

3 Dirac Structures on Tangent Bundles

Definition 3.1. [2] Let M be a smooth n-manifold. A *Dirac structure on the tangent bundle* TM is a maximally isotropic subbundle L of the Whitney sum bundle $TM \oplus T^*M$ under the pairing

$$< (X, \omega), (Y, \mu) >_+ = \frac{1}{2}(\mu(X) + \omega(Y))$$

for $X, Y \in \Gamma(TM)$ and $\omega, \mu \in \Gamma(T^*M)$

Remark 3.2. Let $\tau : TM \oplus T^*M \to T^*M \oplus TM$ be the flip strong bundle isomorphism defined by $\tau(X, \omega) = (\omega, X)$ for $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$. Furthermore let $X = (X_1, ..., X_n)$ and $\omega = (\omega_1, ..., \omega_n)$ be respectively the local coordinate functions of X, ω in a coordinate system on M. The identification between the tangent and cotangent bundles explained in example 2.1 shows that to each $X \in \Gamma(TM)$ there corresponds its dual $\omega_X \in \Gamma(T^*M)$ having the same coordinates as X. Also to each $\omega \in \Gamma(T^*M)$ there corresponds an $X_\omega \in \Gamma(TM)$ having the same coordinate functions as ω.

With these conventions we have

Proposition 3.3. *The subbundle $L \subset TM \oplus T^*M$ is a Dirac structure on TM if and only if L and $\tau(L)$ are ortho-complementary.*

Proof. Suppose L is a Dirac structure on TM. For $(X = (X_i)_i, \omega = (\omega_i)_i) \in L \cap \tau(L)$, we have $((X_1, ..., X_n), (\omega_1, ..., \omega_n)), ((\omega_1, ..., \omega_n), (X_1, ..., X_n)) \in L$ and since L is isotropic, this implies that for each $i = 1, ..., n$, $X_i = \omega_i = 0$. So $L \cap \tau(L) = 0$. Also since L is maximally isotropic, L and $\tau(L)$ are orthogonal and $L \oplus \tau(L) = TM \oplus T^*M$.

Conversely if L and $\tau(L)$ are ortho-complementary, then obviously L is maximally isotropic with respect to the pairing $<, >_+$.

$\square$

Remark 3.4. Let P_1, P_2 be respectively the first and second projections on $TM \oplus T^*M$. Let $L \subset TM \oplus T^*M$ be a Dirac structure on TM. since L is a Dirac structure, then for $(X, \omega), (Y, \beta) \in L$,

$$< P_1(X, \omega), P_2(Y, \beta) > + < P_2(X, \omega), P_1(Y, \beta) >= 0$$

In particular this is true for the basis elements of L, so it implies $P_1 P_2^* + P_2 P_1^* = 0$.

Proposition 3.5. *If $L \subset TM \oplus T^*M$ is a Dirac structure on TM, then the restriction of $P_1 + P_2$ and $P_1 - P_2$ to L are strong bundle isomorphisms.*

Proof. For $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$ with local coordinates $X = (X_i)_i, \omega = (\omega_i)_i$, $i = 1, ..., n$, if $(P_1 + P_2)(X, \omega) = 0$, $X_i = -\omega_i$ for each $i = 1, ..., n$. So $(X, \omega) \in L \cap \tau(L) = 0$, since L is a Dirac structure. Thus $P_1 + P_2$ is injective. The same argument shows that $P_1 - P_2$ is injective.

Let us identify TM with T^*M via the Hermitian inner product. let η be the trivial Hermitian vector bundle of rank n over M, $\mathcal{A}$ the C*-algebra of continuous functions on M and H the Hilbert $\mathcal{A}$-module of sections of η. For each $f = (f_1, ..., f_n) \in H$, let $X \in \Gamma(TM)$ and $\omega \in \Gamma(T^*M)$ both have f as their local coordinate functions. Since L and $\tau(L)$ are ortho-complementary, there are $(Y, \beta) \in L$ and $(Z, \mu) \in \tau(L)$ with local coordinates $Y = (Y_i)_i$, $Z = (Z_i)_i$, $\beta = (\beta_i)_i$ and $\omega = (\omega_i)_i$, such that

$$(X, \omega) = (Y, \beta) \oplus (Z, \mu)$$

Thus $f_i = Y_i + Z_i = \beta_i + \mu_i$ and $Y_i - \mu_i = \beta_i - Z_i$ for all i. So $((Y_i - \mu_i)_i, (Y_i - \mu_i)_i) = ((Y_i - \mu_i)_i, (\beta_i - Z_i)_i) = (Y_i, \beta_i)_i - (\mu_i, Z_i)_i \in L \cap \tau(L) = 0$. Then $Y_i = \mu_i, \beta_i = Z_i$ for all i. And so $X = (f_1, ..., f_n) = (P_1 + P_2)(Y, \beta)$, means that $P_1 + P_2$ is surjective. In the same way we can see that $P_1 - P_2$ is surjective. $\square$

Remark 3.6. With the notations of the previous proposition, if $Aut(TM)$ be the group of strong bundle automorphisms of the bundle TM, then $A = (P_1 + P_2)(P_1 - P_2)^{-1} \in Aut(TM)$ (after the identification of TM with T^*M. Also by the restriction of P_1, P_2 on the sections, we can interpret $A \in Aut(\Gamma(TM))$.

Lemma 3.7. *With the notations of the previous remark, $A \in Aut(\Gamma(TM))$ is orthogonal.*

Proof.

$$AA^* = (P_1 + P_2)(P_1 - P_2)^{-1}(P_1^* - P_2^*)^{-1}(P_1^* + P_2^*)$$

$$= (P_1 + P_2)((P_1^* - P_2^*)(P_1 - P_2))^{-1}(P_1^* + P_2^*)$$

$$= (P_1 + P_2)(P_1^*P_1 - P_1^*P_2 - P_2^*P_1 + P_2^*P_2)^{-1}(P_1^* + P_2^*)$$

$$= (P_1 + P_2)(P_1^*P_1 + P_1^*P_2 + P_2^*P_1 + P_2^*P_2)^{-1}(P_1^* + P_2^*)$$

$$= (P_1 + P_2)((P_1^* + P_2^*)(P_1 + P_2))^{-1}(P_1^* + P_2^*) = I$$

Where we have used the fact in remark 3.4 that $P_1^*P_2 + P_2^*P_1 = 0$.

$\square$

Proposition 3.8. *With the notations of the remark 3.2, let $B \in Aut(\Gamma(TM))$ be orthogonal, then*

$$L_B = \{((I + B)X, (I - B)\omega_X); X \in \Gamma(TM)\}$$

is a Dirac structure on TM.

Proof. Let $((I+B)X,(I-B)\omega_X),((I+B)Y,(I-B)\omega_Y)\in L_B$. Then from example 2.1 and remark 3.2, we have the following equations

$$<\omega_Y, BX>=<B\omega_X, Y>, <\omega_X, BY>=<B\omega_Y, X>$$

and also since B is orthogonal,

$$<\omega_Y, X>=<B\omega_Y, BX>, <\omega_X, Y>=<B\omega_X, BY>$$

These equations imply

$$<(I-B)\omega_Y,(I+B)X>+((I-B)\omega_X,(I+B)Y>=0$$

and so L_B is isotropic.

Now if $(Z,\alpha)\in\Gamma(TM)\oplus\Gamma(T^*M)$ be such that $L_B\cup\{(Z,\alpha)\}$ is isotropic, then for each $((I+B)X,(I-B)\omega_X)\in L_B$, we have

$$0=<((I+B)X,(I-B)\omega_X),(Z,\alpha)>_+$$

$$=<\alpha,(I+B)X>+<(I-B)\omega_X, Z>$$

$$=<\alpha, X>+<\alpha, BX>+<\omega_X, Z>-<B\omega_X, Z>$$

$$=<\alpha, X>+<\alpha, BX>-<\omega_Z, BX>+<\omega_Z, X>$$

$$=<\alpha+\omega_Z, X>+<\alpha-\omega_Z, BX>$$

And so $<B(\alpha+\omega_Z), BX>+<\alpha-\omega_Z, BX>=0$. Thus $B(\alpha+\omega_Z)=\omega_Z-\alpha$. In the same way $B(Z+Z_\alpha)=Z-Z_\alpha$.

where $Z_\alpha\in\Gamma(TM),\omega_Z\in\Gamma(T^*M)$ are respectively the corresponded duals to α and Z as in remark 3.2.

So $Z=\frac{1}{2}(I+B)(Z+Z_\alpha)$ and $\alpha=\frac{1}{2}(I-B)(\alpha+\omega_Z)$. Thus $(Z,\alpha)\in L_B$ and L_B is maximal.

□

Proposition 3.9. *Any Dirac structure $L \subset TM \oplus T^*M$ on TM is of the form L_B for some $B \in Aut(\Gamma(TM))$.*

Proof. We have seen that if L is a Dirac structure on TM, then the restrictions of $P_1 + P_2, P_1 - P_2$ to L are isomorphisms and so $A = (P_1 + P_2)(P_1 - P_2)^{-1} \in Aut(TM)$ is orthogonal. Now to this A there corresponds a $B \in Aut(\Gamma(TM))$ which is orthogonal and so L is the Dirac structure L_B corresponded to B.

□

4 The Topology of Integrable Dirac Structures

Definition 4.1. [1] Let $L \subset TM \oplus T^*M$ be a Dirac structure on TM. Then L is said to be *integrable* if for each $(X, \omega), (Y, \mu) \in L$, we have

$$([X, Y], \{\omega, \mu\}) \in L$$

where $\{\omega, \mu\} = X(d\mu) - Y(d\omega) + \frac{1}{2}d(X(\mu) - Y(\omega))$.

When L is a Dirac structure on TM, in proposition 3.9 we have shown that, L is of the form L_B for some orthogonal $B \in Aut(\Gamma(TM))$. We have the following definition

Definition 4.2. For orthogonal $B \in Aut(\Gamma(TM))$, the Dirac structure L_B is *integrable* if for each pair $((I+B)X, (I-B)\omega_X), ((I+B)Y, (I-B)\omega_Y) \in L_B$, we have

$$(I + B)\{(I - B)\omega_X, (I - B)\omega_Y\} = (I - B)[(I + B)X, (I + B)Y]$$

$B \in Aut(\Gamma(TM))$ is called *integrable automorphism* if L_B is an integrable Dirac structure.

By a straight forward calculation we can see

Lemma 4.3. *The above two definitions for the integrability of Dirac structures are equivalent.*

Corollary 4.4. *For nonzero real number r, $L_{\pm rI}$ are integrable only if $r = \pm 1$.*

Proof. Since for orthogonal $B \in Aut(\Gamma(TM))$ the eigenvalues of B are only ± 1, so $\pm rI \in Aut(\Gamma(TM))$ for $r \neq 1$ are not integrable. □

Now let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^2$ the Euclidean space with the two coordinate functions x, y, R the $\mathbb{R}$-ring of degree two polynomials in x, y and $M = \Gamma(T\mathbb{R}^2)$ the R-module of vector fields on $\mathbb{R}^2$.

Proposition 4.5. *$Aut(M)$ is in one to one correspondence with $GL_2(\mathbb{R}) \times \mathbb{R}^8$.*

Proof. If $A = (a_{ij})_{i,j=1,2} \in Aut(M)$ and $a_{ij} = a^0_{ij} + a^1_{ij}x + a^2_{ij}y$, then $detA$ is invertible. On the other hand

$$detA = a^0_{11}a^0_{22} - a^0_{12}a^0_{21} + ...$$

is invertible iff $a^0_{11}a^0_{22} - a^0_{12}a^0_{21} = 1$.

So the map

$$\theta : Aut(M) \to GL_2(\mathbb{R}) \times \mathbb{R}^8$$

defined by

$$\theta(a_{ij})_{i,j=1,2} = ((a^0_{ij})_{i,j=1,2,...}, ...)$$

is one to one and onto. □

With the notations of the previous proposition, a modification of the definition of the Dirac structure L_A for $A \in Aut(M)$ where $M = \Gamma(T\mathbb{R}^2)$ is as follows

Definition 4.6. If $A \in Aut(M)$, $\theta(A) = (A_0, A_1)$, then

$$L_{A_0} = \{(X + A_0X, X - A_0X); X \in M\}$$

is called *a Dirac structure on M*.

For simplicity we denote L_{A_0} by L_A.

From the proposition 4.5 it follows that each $A \in Aut(M)$ can be considered as an element of $GL_2(\mathbb{R})$. With this convention:

The set of all integrable automorphisms with the norm defined by

$$\| A \|_\infty = sup_{p\in\mathbb{R}^2}\{\| A(p) \|^2 + \sum_{i=1,2} \| \partial_i A(p) \|^2\}^{\frac{1}{2}}$$

for each integrable $A \in Aut(M)$, is a topological group. This group is denoted by $I_D(M)$.

Proposition 4.7. *If $A \in Aut(M)$ is integrable, and if $A \neq -I$, then there exists a curve connecting A to I.*

Proof. For $t \in [0, 1]$, define

$$f : [0, 1] \to Aut(M)$$

by

$$f(t) = \frac{(1-t) + (1+t)A}{(1+t) + (1-t)A}$$

f is continuous and $f(0) = I, f(1) = A$.

□

Proposition 4.8. *$I_D(M)$ has the following properties,*

i)$I_D(M)$ is Hausdorff.

ii)$I_D(M)$ is not connected.

iii)$I_D(M)$ is closed in $O(2)$.

iv)$I_D(M)$ is not open.

Proof. i,ii)$A \in Aut(M)$ is orthogonal, so $I_D(M) \subset O(2)$ and so it is Hausdorff. From corollary 4.4, it follows that $I_D(M)$ has two components, one contains I and the other contains $-I$.

iii)The derivative map is continuous, from the definition of the norm on $I_D(M)$, we see that $I_D(M)$ is closed.

iv)$-I$ is integrable and is the isolated point of $I_D(M)$, so $I_D(M)$ is not open.

□

Set $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$.

Proposition 4.9. *A necessary and sufficient condition for $A = (a_{uv})_{u,v=1,2} \in Aut(M)$ to be integrable is that for $m, i, k = 1, 2$, A satisfies the following differential equation*

$$\partial_i(a_{mk})-\partial_k(a_{mi})+\sum_{l=1,2}(a_{li}\partial_l(a_{mk})-a_{lk}\partial_l(a_{mi}))+\sum_{j=1,2}a_{ji}\partial_m(a_{jk})+\sum_{l,j=1,2}a_{mj}a_{lj}\partial_j(a_{lk}) = 0.$$

Proof. Set $(I - A)dx_i = \alpha$ and $(I - A)dx_k = \beta$. Then using the notations of the Remark 3.2,

$$\alpha = -\sum_{j=1,2} a_{ji}dx_j + dx_i$$

$$\beta = -\sum_{l=1,2} a_{lk}dx_l + dx_k$$

So

$$X_\alpha = -\sum_{j=1,2} a_{ji}\partial_j + \partial_i$$

$$X_\beta = -\sum_{l=1,2} a_{lk}\partial_l + \partial_k$$

So A is integrable iff for $i,k = 1,2$,

$$(I+A)\{\alpha,\beta\} = (I-A)[X_\alpha, X_\beta]$$

On the other hand

$$X_\alpha(\beta) = \beta(X_\alpha) = \sum_{j=1,2} a_{jk}a_{ji} - a_{ik} - a_{ki} + \delta_{ki}$$

$$X_\beta(\alpha) = \alpha(X_\beta) = \sum_{l=1,2} a_{li}a_{lk} - a_{ki} - a_{ik} + \delta_{ik}$$

Also we can write

$$d\alpha = -\sum_{t=1,2}\sum_{j=1,2} \partial_t(a_{ji})dx_t dx_j$$

$$d\beta = -\sum_{t=1,2}\sum_{l=1,2} \partial_t(a_{lk})dx_t dx_l$$

So

$$X_\alpha(d\beta) = d\beta(X_\alpha) =$$

$$\sum_{j,l=1,2} \partial_j(a_{lk})a_{ji}dx_l - \sum_{t,j=1,2} \partial_t(a_{jk})a_{ji}dx_t - \sum_{l=1,2} \partial_i(a_{lk})dx_l + \sum_{t=1,2} \partial_t(a_{ik})dx_t$$

in the same way

$$X_\beta(d\alpha) = d\alpha(X_\beta)$$

$$\sum_{j,l=1,2} \partial_l(a_{jk})a_{lk}dx_j - \sum_{t,l=1,2} \partial_t(a_{li})a_{lk}dx_t - \sum_{j=1,2} \partial_k(a_{ji})dx_j + \sum_{t=1,2} \partial_k(a_{ti})dx_t$$

So

$$\{\alpha, \beta\} = X_\alpha(d\beta) - X_\beta(d\alpha) + \frac{1}{2}d(X_\alpha(\beta) - X_\beta(\alpha))$$

$$= \sum_{l=1,2} (\sum_{j=1,2} \partial_j(a_{lk})a_{ji} - \sum_{j=1,2} \partial_j(a_{li})a_{jk}dx_l) - \sum_{l=1,2} (\sum_{j=1,2} \partial_l(a_{jk})a_{ji}$$

$$- \sum_{j=1,2} \partial_l(a_{ji})a_{jk}dx_l) - \sum_{l=1,2} (\partial_i(a_{lk}) - \partial_k(a_{li})dx_l) \qquad \dagger$$

On the other hand

$$X_\alpha(X_\beta) = \sum_{l,j=1,2} a_{ji}\partial_j(a_{lk})\partial_l - \sum_{l=1,2} \partial_i(a_{lk})\partial_l$$

$$X_\beta(X_\alpha) = \sum_{l,j=1,2} a_{lk}\partial_l(a_{ji})\partial_j - \sum_{j=1,2} \partial_k(a_{ji})\partial_j$$

So

$$[X_\alpha, X_\beta] = X_\alpha(X_\beta) - X_\beta(X_\alpha)$$

$$= \sum_{l,j=1,2} (a_{ji}\partial_j(a_{lk}) - a_{jk}\partial_j(a_{li}))\partial_l) - \sum_{l=1,2} (\partial_i(a_{lk}) - \partial_k(a_{li}))\partial_l \qquad \ddagger$$

Now if we multiply both sides of the equation † by $(I + A)$ and the equation ‡ by $(I - A)$, and then setting them equal to each other we obtain the following differential equation as the necessary and sufficient condition for the integrability of A

$$\partial_i(a_{mk})-\partial_k(a_{mi})+\sum_{l=1,2}(a_{li}\partial_l(a_{mk})-a_{lk}\partial_l(a_{mi}))+\sum_{j=1,2}a_{ji}\partial_m(a_{jk})+\sum_{l,j=1,2}a_{mj}a_{lj}\partial_j(a_{lk})=0.$$

□

Remark 4.10. In the case of the n-dimensional space $\mathbb{R}^n$, we obtain the following differential equation for $m,i,k=1,2,...,n$

$$\partial_i(a_{mk})-\partial_k(a_{mi})+\sum_{l=1,2}(a_{li}\partial_l(a_{mk})-a_{lk}\partial_l(a_{mi}))+\sum_{j=1,2}a_{ji}\partial_m(a_{jk})+\sum_{l,j=1,2}a_{mj}a_{lj}\partial_j(a_{lk})=0.$$

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