MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **89** (2023) 653–664
ISSN: 0340–6253
doi: 10.46793/match.89-3.653A

Some Relations Between Rank, Vertex Cover Number and Energy of Graph

Saieed Akbari[a,*], Hamideh Alizadeh[b], Mohammadhossein Fakharan[a], Mohammad Habibi[b], Samane Rabizadeh[b], Soheyr Rouhani[b]

[a]*Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran.*

[b]*Department of Mathematics, Tafresh University, Tafresh 39518-79611, Iran.*

s_akbari@sharif.edu, hamidehh.alizadehh@gmail.com, mh.fakharan92@sharif.edu, mhabibi@tafreshu.ac.ir, samane.rabizadeh@gmail.com, sorouhani435@gmail.com

(Received November 8, 2022)

Abstract

In this paper, we extend some results of [F. Shaveisi, lower bounds on the vertex cover number and energy of graphs, MATCH Commun. Math. Comput. Chem, 87(3) (2022) 683-692] which state some relations between the vertex cover and other parameters, such as the order and maximum or minimum degree of graphs. Also, we prove that for a graph G, $\mathcal{E}(G) \geq 2\beta(G) - 2C_e(G)$ and so $\mathcal{E}(G) \geq 2\beta(G) - 2C(G)$, where $\mathcal{E}(G)$, $\beta(G)$, $C_e(G)$ and $C(G)$ denote the energy, vertex cover, number of even cycles and number of cycles in G, respectively. For these both inequalities we investigate their equality. Finally, we give some relations between $\mathcal{E}(G), \gamma(G)$ and $\gamma_t(G)$, where $\gamma(G)$ and $\gamma_t(G)$ are domination number and total domination number of G, respectively.

*Corresponding author.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the set of its vertices and edges, respectively. By the size of G, we mean the number of its edges. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The *adjacency matrix* of G, denoted by $A(G)$, is an $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent and 0 otherwise. The *corona* of two graphs, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 (which has n vertices) and n copies of G_2, and then joining the ith vertex in G_1 to every vertex in the ith copy of G_2. In this paper, the *energy* of a graph G, is shown by $\mathcal{E}(G)$ and is defined as the sum of the absolute values of its adjacency eigenvalues. A *vertex cover* of a graph is a subset of vertices that includes at least one endpoint of every edge of the graph. The minimum size of a vertex cover of G is called the *vertex cover number* and is denoted by $\beta(G)$. The number of connected components of G is denoted by $c(G)$, and we define $cv(G) = \min \{c(G[Q]) : Q$ is a minimum vertex cover of $G\}$. Also, for a set $Q \subset V(G)$, $G[Q]$ means the induced subgraph of G on Q. For a graph G, $C_o(G)$ and $C_e(G)$ denote the number of odd and even cycles in G, respectively. The number of all cycles in G is denoted by $C(G)$. A *dominating set* in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S and a *total dominating set* of G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *domination number* (*total domination number*) of G, denoted by $\gamma(G)$ ($\gamma_t(G)$), is the minimum cardinality of a dominating set (a total dominating set) of G. A maximum matching is a matching that contains the largest possible number of edges. If a matching covers all vertices of G, then it is called a perfect matching. The *matching number* of G, denoted by $\mu(G)$, is the size of a maximum matching. We denote the complete graph and the cycle graph of order n by K_n and C_n, respectively.

In all of the above notation, we remove the additional G if there is no ambiguity; for example δ instead of $\delta(G)$, or V instead of $V(G)$.

2 Preliminaries

In the following, we state some lemmas which are used in our proofs.

Lemma 1. [2]. *Let G be a graph and $H_1, \ldots, H_k$ be its k vertex-disjoint induced subgraphs. Then* $\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i)$.

Lemma 2. [1, Lem. 11]. *If n is an odd integer, then* $\mathcal{E}(C_n) \geq n+1$.

Lemma 3. [11, Thm. 1.1]. *Let G be a graph. Then* $\mathcal{E}(G) \geq 2\mu(G)$.

Lemma 4. [6]. *If G is a graph without any isolated vertex, then* $\mu(G) \geq \gamma(G)$.

Lemma 5. [5, Thm. 3]. *A connected graph G of order $2n$ has $\gamma(G) = n$ if and only if either $G = C_4$ or the vertices of G can be partitioned into two sets, V_1 and V_2 with a matching between them and satisfying $G[V_1] = \overline{K_n}$ and $G[V_2]$ connected.*

Lemma 6. [7, Thm. 4.20]. (1) *If F is an edge cut of a simple graph G, then $\mathcal{E}(G-F) \leq \mathcal{E}(G)$.* (2) *Let H be a subgraph of G and F be the edge cut between $G-H$ and H. Suppose that F is not empty and that all edges in F are incident to one and only one vertex in H, i.e. the edges in F form a star. Then $\mathcal{E}(G-F) < \mathcal{E}(G)$.*

3 Main results

We start this section by the following theorem that extends Theorems 1 and 2 of [8] by considering the values of δ, i.e. we extend the results if $\delta \geq k$. If we put $k=1$, then both Theorems 1 and 2 of [8] are an immediate consequence of the following theorem.

Theorem 1. *Let G be a graph of order n with $\delta \geq k$. Then the following hold:*

(i) $\beta > \dfrac{n}{\Delta + 2 - k}$,

(ii) $\beta \geq \dfrac{kn - 2cv(G)}{\Delta + k - 2}$.

Proof. First, we claim that $n \leq \beta\Delta+\beta-(k-1)(n-\beta)$. Clearly, $n \leq \beta\Delta+\beta$. Assume that Q is a covering set of order β. Suppose that $v \in V(G) \setminus Q$. Since $G \setminus Q$ is an independent set, $|N_Q(v)| \geq k$. Without lose of generality, assume that $v_1, \ldots, v_k \in Q$ are adjacent to v. In this case,

$$|N(v_1) \cup \cdots \cup N(v_k)| \leq k\Delta - (k-1).$$

Hence, each vertex $v \in G \setminus Q$ decreases the bound $\beta\Delta+\beta$ at least by $k-1$. Thus, $n \leq \beta\Delta+\beta-(k-1)(n-\beta)$ and the claim is proved. Now, we claim that if there exist t edges in $G[Q]$, then $n \leq \beta\Delta+\beta-(k-1)(n-\beta)-2t$. For this, suppose that u and v in Q are adjacent. Therefore the number of vertices in $G \setminus Q$ that are adjacent to u or v is at most $2\Delta-2$. This means that each edge in $G[Q]$ decreases the upper bound $\beta\Delta+\beta-(k-1)(n-\beta)$ by 2 and thus the second claim is proved.

For Part (i), if $\beta > \frac{n}{2}$, then clearly $\beta > \frac{n}{\Delta+2-k}$, since $\Delta - k \geq 0$. So suppose $\beta \leq \frac{n}{2}$ and by contrary $\beta \leq \frac{n}{\Delta+2-k}$. Therefore, by the first claim we have

$$\beta < \frac{\beta\Delta+\beta-(k-1)(n-\beta)}{\Delta+2-k}$$

and consequently $\beta > \frac{k}{2k-1}n > \frac{n}{2}$, a contradiction.

For Part (ii), let Q be a minimum vertex cover of the graph G in which $c(G[Q]) = cv(G)$. Suppose the i^{th} connected component of $G[Q]$ has order β_i, for $i = 1, \ldots, cv(G)$. So it has at least $\beta_i - 1$ edges and hence by the second claim, one can see that

$$n \leq \beta\Delta+\beta-(k-1)(n-\beta)-\sum_{i=1}^{cv(G)} 2(\beta_i-1) = \beta\Delta-\beta-(k-1)(n-\beta)+2cv(G),$$

which yields that $\beta \geq \frac{kn-2cv(G)}{\Delta+k-2}$ and the proof is complete. ∎

Remark 1. *Let G be a connected graph of size m and $\Delta \geq 2$. In Theorem 5 of [8], with a long proof, it is proved that*

$$\beta \geq \frac{\sqrt{(2\Delta-1)^2+8m}-(2\Delta-1)}{2}.$$

Clearly, $m \leq \beta\Delta$ *and so* $\beta \geq \frac{m}{\Delta}$. *By some calculations, it is easy to see that* $2m + 2\Delta^2 - \Delta \geq \Delta\sqrt{(2\Delta-1)^2 + 8m}$. *Hence*

$$\frac{m}{\Delta} \geq \frac{\sqrt{(2\Delta-1)^2 + 8m} - (2\Delta - 1)}{2}.$$

Thus $\frac{m}{\Delta}$ *is a better bound for* β. *Also, since* $\frac{m}{\Delta} \geq \frac{n}{\Delta+1}$, *Corollary 6 of [8] cannot give us new information. In addition by Theorem 4.2 of [9], there is a much better lower bound* $2(\frac{m}{\Delta} - c_o)$ *for the energy of a graph instead of what is introduced in [8, Cor. 10]. Surprisingly, there is no any difference between Corollaries 10 and 12 of [8]. Furthermore, Lemma 11 of [8] is presented just for clarifying Corollary 12 which is equal to Corollary 10.*

The next theorem is proved by Chen and Liu in [4] (Proposition 6), but here we give an easier and shorter proof.

Theorem 2. *Let* G *be a graph of order* n *with the adjacency matrix* A. *Then* $rank(A) \leq 2\beta$.

Proof. Let $Q = \{v_1, \ldots, v_\beta\}$ be a minimum vertex cover of G. With an appropriate labeling for vertices, we have $A = \begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix}$, where B is the adjacency matrix of $G[Q]$. Obviously, in the first β rows of A, there are maximum β independent rows. Also, $\text{rank}(C^T) \leq \beta$ and so the inequality follows. ∎

In the following theorem, we state a sufficient condition so that equality in the above theorem occurs.

Theorem 3. *Let* G *be a graph of order* n *with the adjacency matrix* A. *If* B *is a non-singular* $(0,1)$*-matrix of order* n *and* H *is the graph whose adjacency matrix is* $A' = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$, *then* $rank(H) = 2\beta(H) = 2n$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = V(G) \cup \{u_1, \ldots, u_n\}$. Note that for $i = 1, \ldots, n$, we have $N_H(u_i) = \{v_j \mid b_{ij} = 1\}$. Since $V(G)$ is a vertex cover for H, we conclude that $\beta(H) \leq n$. Now, we show that $\beta(H) \geq n$. To see this, it suffices to prove that H has a perfect matching. Since $H \setminus E(G)$ is a spanning subgraph of H, if we show that $H \setminus E(G)$

has a perfect matching, then we are done. For simplicity call the graph $H \setminus E(G) = H^{'}$. Note that $H^{'} = (U, V(G))$ is a bipartite graph, where $U = \{u_1, \ldots, u_n\}$. By Marriage Theorem [3], it is enough to show that for every $S \subseteq U$, $|N_{H'}(S)| \geq |S|$. By contrary, suppose that there exists $S \subseteq U$ such that $|S| = r$ and $t = |N_{H'}(S)| < r$. With no lose of generality assume that $S = \{u_1, \ldots, u_r\}$. Hence there are $t < r$ rows in B which contain all non-zero entries of B appeared in the first r columns of B. Now, if $B^{'}$ is an $n \times r$ submatrix of B formed by the first r columns of B, then $\text{rank}(B') \leq t$. Thus $\text{rank}(B) \leq t + n - r < n$ and so B is singular, a contradiction. Therefore H' has a perfect matching which implies that $\beta(H) \geq n$ and so $\beta(H) = n$. Also, since B and B^T are non-singular, one can easily see that $A^{'}$ is non-singular. Thus $\text{rank}(H) = \text{rank}(A^{'}) = 2n = 2\beta(H)$ and the result follows. ■

Note that the inverse of previous theorem is true.

Remark 2. *If H is a graph of order $2n$ with the non-singular adjacency matrix A' and rank$(H) = 2\beta(H) = 2n$, then there exists a graph G of order n with adjacency matrix A and a $(0,1)$ non-singular square matrix B of order n such that*

$$A^{'} = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}.$$

To see this, let $S = \{v_1, \ldots, v_n\}$ be a minimum vertex cover for H and $G = H[S]$. Since S is a vertex cover, $V(H) \backslash S$ is an independent set. Obviously, by a suitable labeling of vertices of H, we have $A^{'} = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix}$, where A is the adjacency matrix of G. Since $A^{'}$ is non-singular, the columns of B are linearly independent, that is B is a non-singular $(0,1)$-square matrix of order n.

Wang and Ma [9] provided the following lower bound for the energy of a graph in terms of its cover vertex number and the number of odd cycles.

Theorem 4. [9, Thm. 4.2]. *Let G be a graph with C_o odd cycles. Then $\mathcal{E}(G) \geq 2\beta(G) - 2C_o(G)$, equality holds if and only if G is the disjoint union of some complete bipartite graphs with perfect matchings together with some isolated vertices.*

Now, we show that the above result also holds if one replaces $C_o(G)$ with $C_e(G)$. Moreover, we characterize the equality case. To prove it, we need the following corollary which easily can be deduced from Lemma 6.

Corollary 1. *If H is a proper subgraph of a graph G, then $\mathcal{E}(H) < \mathcal{E}(G)$.*

Theorem 5. *Let G be a graph. Then $\mathcal{E}(G) \geq 2\beta(G) - 2C_e(G)$. Moreover, the equality holds if and only if $C_e(G) = 0$ and G is a disjoint union of some $K_i, (i = 1, 2, 3)$.*

Proof. We prove the inequality by induction on C_e. First suppose that $C_e(G) = 0$. In this case, by Exercise 4.2.18 of [10], G has a block decomposition in which every block is K_2 or an odd cycle. By induction on the number of vertices, we show that $\mathcal{E}(G) \geq 2\beta(G)$. For $n = 2$, the assertion is trivial. If there exists a pendant vertex v adjacent to a vertex, say u, then by induction hypothesis $G \setminus \{u, v\} = G'$ has energy at least $2\beta(G')$. Note that the union of a vertex cover for G' and $\{u\}$ is a vertex cover for G. So by Lemma 1 one can see that $\mathcal{E}(G) \geq \mathcal{E}(G') + 2 \geq 2\beta(G') + 2 \geq 2(\beta(G) - 1) + 2 = 2\beta(G)$. Now, suppose that there is no pendant vertex. Therefore there is a leaf block C_{2k+1} containing a unique cut vertex of G, say x. Let $G' = G \setminus V(C_{2k+1})$. By induction hypothesis and Lemmas 1 and 2, $\mathcal{E}(G) \geq \mathcal{E}(G') + \mathcal{E}(C_{2k+1}) \geq 2\beta(G') + 2k + 2$. By considering a vertex cover of size $k + 1$ of C_{2k+1} containing x, it is easy to see that $\beta(G') + k + 1 \geq \beta(G)$. Thus $\mathcal{E}(G) \geq 2\beta(G)$. Now, suppose that the inequality holds for the graphs with at most $C_e(G) - 1$ even cycles, and G is a graph with $C_e(G) \geq 1$ even cycles. Let x be a vertex of G lying on an even cycle. Thus $G - x$ has at most $C_e(G) - 1$ even cycles. Thus the induction hypothesis implies that $\mathcal{E}(G - x) \geq 2\beta(G - x) - 2(C_e(G) - 1)$. Since $\beta(G - x) + 1 \geq \beta(G)$, then by Corollary 1, we have

$$\mathcal{E}(G) > \mathcal{E}(G - x) \geq 2\beta(G - x) - 2(C_e(G) - 1) \geq 2\beta(G) - 2C_e(G)$$

and the inequality is proved.

Now, suppose that the equality holds. So $C_e(G) = 0$. With no lose of generality suppose that G is connected of order n. Using induction on n, we show that G is K_i, $(i = 1, 2, 3)$. If $n = 1$, then $G = K_1$. Now, assume that the result holds for all graphs of order less than n and that $|V(G)| = n$.

Case 1. There is a leaf block C_{2k+1} with a unique cut vertex x of G. Let $G^{'} = G \setminus V(C_{2k+1})$. By Lemmas 1 and 2 and the induction hypothesis

$$\mathcal{E}(G) \geq \mathcal{E}(G^{'}) + 2k + 2 \geq 2\beta(G^{'}) + 2k + 2 \geq 2\beta(G) = \mathcal{E}(G). \qquad (*)$$

Thus by induction hypothesis every connected component of G' is K_1, K_2 or K_3. If K_2 is one of the connected components of G' with vertices u and v, then by Lemma 1 and induction hypothesis for $G'' = G \setminus \{u, v\}$ we have a similar inequalities as $(*)$ where $k = 0$. Note that G'' is connected and so by induction hypothesis G'' is K_3 and so G is one of the graphs shown in Figure 1 (i) and (ii) which is not satisfy the equality, a contradiction. If K_3 is one of the connected components of G' with vertices u, v and w, then by Lemma 1 and induction hypothesis for $G'' = G \setminus \{u, v, w\}$ we have a similar inequalities as $(*)$ where $k = 1$. Note that G'' is connected and so by induction hypothesis G'' is K_3 and so G is the graph shown in Figure 1 (iii) which is not satisfy the equality, a contradiction. If all connected components of G' are K_1, then G is the graph shown in Figure 1 (iv) (i.e. a cycle C_{2k+1} whose one vertex is adjacent to some pendant vertices) which is not satisfy the equality because by Corollary 1 and Lemma 2, one can see that $\mathcal{E}(G) > \mathcal{E}(C_{2k+1}) \geq 2k + 2 = 2\beta(G)$. Hence $G = C_{2k+1}$ and due to $\mathcal{E}(G) = 2\beta(G)$, $G = K_3$.

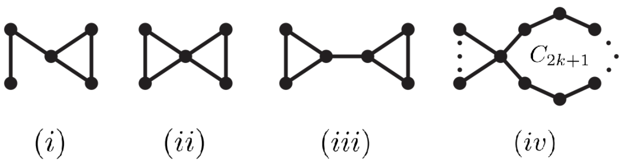

Figure 1

Case 2. All of leaf blocks are K_2. If $n = 2$, then $G \cong K_2$. Now, let $n \geq 3$ and v is a pendent vertex adjacent to u. Suppose $G' = G \setminus \{u, v\}$. So by relations $(*)$, every connected components of G' satisfies induction hypothesis and so equals to K_1 or K_2. In this case, it is not hard to see that G is the graph shown in Figure 2. If there exists a vertex w such

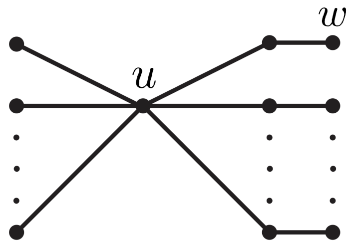

Figure 2

that $d(u, w) = 2$, then remove w and its neighbor and apply induction hypothesis. Therefore the remaining graph is K_1 or K_2. So $G = P_4$ or $G = P_3$ which do not satisfy the equality. Thus suppose that G is a star and so clearly, $G = K_2$.

Conversely, if every component of G is a complete graph of order at most 3, then clearly $C_e(G) = 0$ and $\mathcal{E}(G) = 2\beta(G)$; and the proof is complete. ■

The interesting point is that in both Theorems 4 and 5, the necessary condition for the equality is that C_o and C_e must be zero. So, K_1 and K_2 are the only graphs that satisfy both equalities in these two theorems.

By combining two previous theorems, we can deduce the next important result about the relation between $\mathcal{E}(G), \beta(G)$ and $C(G)$.

Corollary 2. *Let G be a graph. Then $\mathcal{E}(G) \geq 2\beta(G) - C(G)$. Moreover, equality holds if and only if G is a disjoint union of some K_1 or K_2.*

Proof. By adding two inequalities of Theorems 4 and 5 and paying attention to the fact that $C(G) = C_o(G) + C_e(G)$, the desired inequality is obtained. Now, suppose that $\mathcal{E}(G) = 2\beta(G) - C(G)$. There are two following cases:

Case 1. $C_e(G) \leq C_o(G)$. In this case, by Theorem 5, one can easily see that $\mathcal{E}(G) = 2\beta(G) - C(G) \leq 2\beta(G) - 2C_e(G) \leq \mathcal{E}(G)$. So $C(G) = 2C_e(G) = 0$ and hence $C_o(G) = 0$. Thus $G = \bigcup K_i$, $i = 1, 2$.

Case 2. $C_o(G) \leq C_e(G)$. In this case, by Theorem 4, one can easily see that $\mathcal{E}(G) = 2\beta(G) - C(G) \leq 2\beta(G) - 2C_o(G) \leq \mathcal{E}(G)$. So $C(G) = 2C_o(G) = 0$ and G is the disjoint union of some $K_{t,t}$ for integers t together with some isolated vertices. On the other hand, $C_e(G) = 0$ implies that $G = \bigcup K_i$, where $i = 1, 2$.

Finally, if G is a disjoint union of some K_i where $i = 1, 2$, then $\mathcal{E}(G)$ equals twice of the number of K_2 which is equal to $2\beta(G)$. So equality holds and the proof is complete. ■

According to Theorems 4 and 5, our next purpose is to remove the coefficient 2 of $C_o(G)$ and $C_e(G)$. To achieve this goal, investigating the next conjecture which is true for almost all graphs can be interesting.

Conjecture 1. *Let G be a graph. Then*

$$\mathcal{E}(G) \geq min\{2\beta(G) - C_e(G), 2\beta(G) - C_o(G)\}.$$

Remark 3. *Let G be a graph of order n and size m. Then Conjecture 1 holds if $m \geq 5n$. Because adding t edge to a spanning tree of G makes at least t cycles, one can see that one of the $C_e(G)$ or $C_o(G)$ is at least $2n$ and so $min\{2\beta(G) - C_e(G), 2\beta(G) - C_o(G)\} \leq 0$. Consequently, if $\delta(G) \geq 10$, then the conjecture holds.*

Theorem 6. *Let G be a graph without isolated vertices. Then $\mathcal{E}(G) \geq 2\gamma(G)$. Moreover, equality holds if and only if each connected component of G is K_2 or C_4.*

Proof. Clearly, $\mathcal{E}(G) \geq 2\mu(G) \geq 2\gamma(G)$, by Lemmas 3 and 4. Now, suppose that $\mathcal{E}(G) = 2\gamma(G)$. Hence, $\mathcal{E}(G) = 2\mu(G) = 2\gamma(G)$. If G does not have a perfect matching, then by Corollary 1, $\mathcal{E}(G) > 2\mu(G)$, a contradiction. So assume that G has a perfect matching. Therefore $n = 2\gamma(G)$. Consider one of its connected components, say G'. By Lemma 5, $G' = G_1 \circ K_1$, for a suitable graph G_1, or $G' = C_4$. If G_1 has at least one edge uv, then let H' be the induced subgraph P_4 on u, v and their pendant neighbors and let $H_1, \ldots, H_{|V(G_1)|-2}$ be some K_2. So, $\mathcal{E}(G') > 2\gamma(G')$, by Lemma 1 and considering the fact that $\mathcal{E}(P_4) > 4$; and hence $\mathcal{E}(G) > 2\gamma(G)$, a contradiction. Therefore, $G_1 = K_1$ and consequently each component of G is K_2 or C_4. The converse is obvious and we are done. ■

We finish the paper with the following results about the energy and the total dominating number.

Theorem 7. *Let G be a graph without isolated vertices. Then $\mathcal{E}(G) \geq \gamma_t(G)$.*

Proof. Note that each maximum matching set is a total dominating set for G. So by Lemma 3, $\mathcal{E}(G) \geq 2\mu(G) \geq \gamma_t(G)$. ■

Note that for the path P_5 we have $\mathcal{E}(P_5) < 2\gamma_t(P_5)$. Therefore, it is not correct that $\mathcal{E}(G) \geq 2\gamma_t(G)$, in general.

Remark 4. *The key inequality in the proof of Theorem 7 is $\gamma_t(G) \leq 2\mu(G)$. Hence, investigating this inequality can be interesting. Actually, we cannot improve this inequality. In fact, for every real number $\varepsilon > 0$, there is a graph G with $\gamma_t(G) > (2-\varepsilon)\mu(G)$. For constructing such a graph, consider the graph G shown in the Figure 3. For this graph, we have $\gamma_t(G) = 2t+1$, $\mu(G) = t+1$ and so $\frac{\gamma_t(G)}{\mu(G)} = \frac{2t+1}{t+1} = 2 - \frac{1}{t+1}$.*
Hence $\lim\limits_{t\to\infty} \frac{\gamma_t(G)}{\mu(G)} = 2$.
Therefore, for every $\varepsilon > 0$, if we put $t > \frac{1}{\varepsilon} - 1$, then we have $\gamma_t(G) > (2-\varepsilon)\mu(G)$.

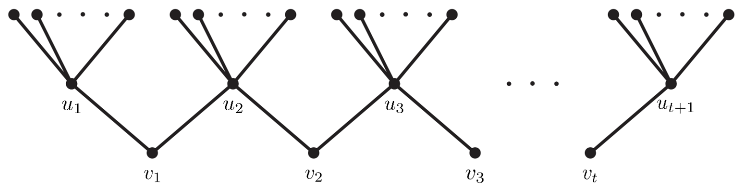

Figure 3

***Acknowledgment*:** The research of the first author was supported by grant number G981202 from Sharif University of Technology. Also, part of the work in this paper was done while the third author was at Concordia University.

References

[1] S. Akbari, M. Ghahremani, I. Gutman, F. Koorepazan-Moftakhar, Orderenergetic graphs, *MATCH Commun. Math. Comput. Chem.* **84** (2020) 325–334.

[2] S. Akbari, E. Ghorbani, M. R. Oboudi, Edge addition, singular values, and energy of graphs and matriced, *Lin. Algebra Appl.* **430** (2009) 2192–2199.

[3] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, London, 2008.

[4] X. Chen, X. Liu, Remarks on the bounds of graph energy in terms of vertex cover number or matching number, *Czechoslov. Math. J.* **71** (2021) 309–319.

[5] J. F. Fink, M. S. Jacobson, L. F. Kinch, J. Roberts, On graphs having domination number half their order, *Period. Math. Hung.* **16** (1985) 287–293.

[6] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[7] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[8] F. Shaveisi, Lower bounds on the vertex cover number and energy of graphs, *MATCH Commun. Math. Comput. Chem.* **87** (2022) 683–692.

[9] L. Wang, X. Ma, Bounds of graph energy in terms of vertex cover number, *Lin. Algebra Appl.* **517** (2017) 207–216.

[10] D. B. West, *Introduction to Graph Theory*, Prendice Hall, Upper Saddle River, 2001.

[11] D. Wong, X. Wang, R. Chu, Lower bounds of graph energy in terms of matching number, *Lin. Algebra Appl.* **549** (2018) 276–286.