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Synchronization of chaotic Lur'e systems with state and transmission line time delay: a linear matrix inequality approach

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Abstract

This paper is concerned with the problem of the master–slave synchronization of chaotic Lur'e systems with multiple time delays in their states and transmission line. Based on the Lyapunov–Krasovskii functional, some delay-dependent synchronization criteria are obtained and formulated in the form of linear matrix inequalities (LMIs) to ascertain the global asymptotic stability of the error system such that the slave system is synchronized with the master. With the help of the LMI solvers, the time-delay feedback control law can easily be obtained. The effectiveness of the proposed method is illustrated using some numerical simulations performed on two chaotic systems.

Keywords

Chaos, synchronization, Chua's circuit, Lyapunov–Krasovskii functional, time-delay systems

Introduction

Chaos synchronization has been considered by many researchers during the last decade. This is mainly because it has many potential applications in various fields including chaos generators design, secure communications, chemical reactions, biological systems and information science (Ding and Han, 2012; Kwon et al., 2011; Li et al., 2012; Pecora and Carroll, 1990; Zhang et al., 2014). As many non-linear control systems can be represented as Lur'e systems, a unified approach to the chaos synchronization of such systems is to reformulate it as a Lur'e system and then investigate the absolute stability of its error dynamics (Guo and Zhong, 2007). The notion of absolute stability was first introduced by Lur'e (1957). In particular, due to the propagation delay, there has been some research effort to investigate the delay effect on the master–slave synchronization for continuous and discrete systems (Huang and Feng, 2008, 2009a; Kokil et al., 2011; Lee et al., 2010; Liu et al., 2010; Xiang et al., 2007; Zhong and Han, 2009). Cao et al. (2005), Guo and Zhong (2007), Han (2007) and Liu (2010) have considered feedback control for synchronization of time-delay systems. However, they have assumed that the gain matrices for the feedback controller are given. Xiang et al. (2007), Lu and Hill (2008), Huang et al. (2009b), Li et al. (2009), Kwon et al. (2013) and Ge et al. (2014) have proposed some delay-dependent conditions in the form of linear matrix inequalities (LMIs). The gain matrices are obtained by solving the LMIs.

In general, in chaos synchronization problems, the master and slave systems may have time delays in their states. However, most papers in this area assume that these systems do not have a time delay in their states (Cao et al., 2005; Ge et al., 2014; Han, 2007; Huang and Feng, 2008, 2009a, 2009b; Kwon et al., 2013; Lee et al., 2010; Li et al., 2009; Liu et al., 2010; Lu and Hill, 2008; Xiang et al., 2007; Zeng et al., 2015; Zhong and Han, 2009). Guo and Zhong (2007) have assumed a single time delay for the Lur'e systems that is equal to the time delay of the transmission line. This is a conservative assumption.

In this manuscript, the problem of synchronization of chaotic Lur'e systems with multiple time delays using the time-delay feedback control is considered. Based on the Lyapunov–Krasovskii functional approach, some delay-dependent criteria are obtained and formulated in the form of LMIs, which are solved to obtain the state-feedback gains. The main contribution of this paper is considering multiple time delays for the master and slave systems that are not necessarily equal to the time delay of the transmission line. Hence, the results are less conservative in comparison with the existing works as far as the maximum stabilizing propagation delay is concerned. Therefore, a larger class of systems can be considered for the synchronization problem. In

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addition, the gain matrices of the synchronization are obtained automatically by solving the LMIs. The proposed method is applied to two chaotic systems to illustrate the effectiveness of the criteria.

The organization of the paper is as follows. The next section presents the problem formulation and the master–slave synchronization structure for the Lur'e systems with multiple time delays. Then, based on the Lyapunov–Krasovskii functional and the LMI, some delay-dependent criteria are given to ascertain the global asymptotic stability of the error system such that the slave system synchronizes with the master system. The global asymptotic stability condition guarantees that the error asymptotically approaches zero for any initial condition. Some simulation results are provided and finally we conclude the paper.

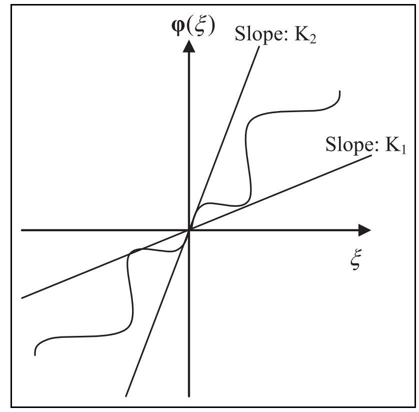

Figure 1. A typical non-linear function with sector constraint.

Notation

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all real $n\times m$ matrices. Moreover, $\mathbf{P}>0$ indicates a real positive definite and symmetric matrix, $\mathbb{C}[-h,0]$ denotes space of continuous functions defined on $[-h,0]$, $\mathbf{I}$ is the identity matrix with appropriate dimensions, $\mathrm{diag}\{w_1,\ \ldots,\ w_m\}$ refers to a real matrix with diagonal elements $w_1,\ \ldots,\ w_m$, and $\mathbf{A}^T$ denotes the transpose of the real matrix $\mathbf{A}$. In addition, symmetric terms in a symmetric matrix are denoted by $*$.

Problem formulation

As mentioned in the introduction, many chaotic systems can be reformulated as a Lur'e system. Hence, consider a general master–slave synchronization arrangement of two chaotic multiple time-delayed Lur'e system as

Master system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{m}\mathbf{A}_i\mathbf{x}(t-h_i) + \mathbf{D}\boldsymbol{\omega}_x(t) \\ \mathbf{z}_x(t) = \mathbf{M}\mathbf{x}(t) \\ \boldsymbol{\omega}_x(t) = -\boldsymbol{\varphi}(\mathbf{z}_x(t)) \\ \mathbf{x}(t) = \boldsymbol{\phi}_x(t), \qquad \forall t\in[-\max\limits_{1\le i\le m}\{h_i\},0] \end{cases} \tag{1}$$

Slave system

$$\begin{cases} \dot{\mathbf{y}}(t) = \mathbf{A}\mathbf{y}(t) + \sum_{i=1}^{m}\mathbf{A}_i\mathbf{y}(t-h_i) + \mathbf{D}\boldsymbol{\omega}_y(t) + \mathbf{u}(t) \\ \mathbf{z}_y(t) = \mathbf{M}\mathbf{y}(t) \\ \boldsymbol{\omega}_y(t) = -\boldsymbol{\varphi}\big(\mathbf{z}_y(t)\big) \\ \mathbf{y}(t) = \boldsymbol{\phi}_y(t), \quad \forall t\in[-\max\limits_{1\le i\le m}\{h_i\},0] \end{cases} \tag{2}$$

with the controller

$$\mathbf{u}(t) = \mathbf{K}_e(\mathbf{x}(t)-\mathbf{y}(t)) - \mathbf{K}_z\big(\mathbf{z}_x(t-h_d)-\mathbf{z}_y(t-h_d)\big) \tag{3}$$

where $\mathbf{x}(t),\ \mathbf{y}(t)\in\mathbb{R}^n$ denote the state vectors, $\mathbf{z}_x(t),\ \mathbf{z}_y(t)\in\mathbb{R}^p$ are the output vectors, $\boldsymbol{\phi}_x(t),\ \boldsymbol{\phi}_y(t)\in\mathbb{C}([h,0],\mathbb{R}^n)$ are the continuous vector-valued initial functions and $\boldsymbol{\varphi}(\mathbf{z}_x(t)),\ \boldsymbol{\varphi}(\mathbf{z}_y(t))\in\mathbb{R}^p$ are time-invariant non-linear functions, which will be defined in Assumption 1. Moreover, $h_i>0$ $(i=1,\ \ldots,\ m)$ are known time delays for the master and slave systems, $h_d>0$ is the known time delay in communication channel and $\mathbf{M}\in\mathbb{R}^{p\times n}$, $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{A}_i\in\mathbb{R}^{n\times n}, (i=1,\ \ldots,\ m)$ and $\mathbf{D}\in\mathbb{R}^{n\times p}$ are known constant matrices of the master and slave systems.

Remark 1. From a practical point of view, a known and constant delay feedback can be realized by introducing a time buffer in the receiver even if the delay is time-varying (Xiang et al., 2007).

Assumption 1. The non-linear function $\boldsymbol{\varphi}(\xi)\in\mathbb{R}^p$ in (1) and (2) is globally Lipschitz in ξ, $\boldsymbol{\varphi}(0)=0$, and satisfies the following sector condition for any $t\ge 0$ and ξ:

$$[\boldsymbol{\varphi}(\xi)-\mathbf{K}_1\xi]^T[\boldsymbol{\varphi}(\xi)-\mathbf{K}_2\xi]\le 0, \qquad \forall\xi \tag{4}$$

Such a non-linear function is said to belong to the known sector $[\mathbf{K}_1,\mathbf{K}_2]$, where $\mathbf{K}_2-\mathbf{K}_1>0$ (Khalil and Grizzle, 1996).

Remark 2. A typical non-linear function $\boldsymbol{\varphi}(\xi)$ satisfying (4) is shown in Figure 1. According to this figure, $\mathbf{K}_1$ and $\mathbf{K}_2$ can take values in $[0,\infty)$. Obviously, the condition $\mathbf{K}_2-\mathbf{K}_1>0$ should be met.

The goal is to design the gain matrices $\mathbf{K}_e\in\mathbb{R}^{n\times n}$ and $\mathbf{K}_z\in\mathbb{R}^{n\times p}$ in (3) to synchronize the slave with the master.

By applying controller (3) to the slave system (2) and defining synchronization error as $\mathbf{e}(t)=\mathbf{x}(t)-\mathbf{y}(t)$, the error dynamical system can be obtained in the following form:

$$\begin{cases} \dot{\mathbf{e}}(t) = (\mathbf{A}-\mathbf{K}_e)\mathbf{e}(t) + \sum_{i=1}^{m+1}\mathbf{A}_i\mathbf{e}(t-h_i) + \mathbf{D}\boldsymbol{\omega}_e(t) \\ \boldsymbol{\omega}_e(t) = -\boldsymbol{\varphi}_e(\mathbf{Me},\mathbf{y}) \\ \boldsymbol{\varphi}_e(\mathbf{Me},\mathbf{y}) = \boldsymbol{\varphi}(\mathbf{Me}+\mathbf{My}) - \boldsymbol{\varphi}(\mathbf{My}) \end{cases} \tag{5}$$

where $\mathbf{A}_{m+1}=\mathbf{K}_z\mathbf{M}$, $h_{m+1}=h_d$ and $\boldsymbol{\varphi}_e(\mathbf{Me},\mathbf{y})$ belongs to the known sector $[\mathbf{K}_1,\mathbf{K}_2]$ (Pecora and Carroll, 1990). Therefore,

$$[\boldsymbol{\varphi}_e(\mathbf{Me},\mathbf{y})-\mathbf{K}_1\mathbf{Me}(t)]^T[\boldsymbol{\varphi}_e(\mathbf{Me},\mathbf{y})-\mathbf{K}_2\mathbf{Me}(t)]\le 0 \tag{6}$$

By applying the loop transformation (Khalil and Grizzle, 1996), (5) can be transformed into the following form:

$$\begin{cases} \dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{D}\mathbf{K}_1\mathbf{M} - \mathbf{K}_e)\mathbf{e}(t) + \sum_{i=1}^{m+1} \mathbf{A}_i\mathbf{e}(t-h_i) + \mathbf{D}\tilde{\boldsymbol{\omega}}_e(t) \\ \tilde{\boldsymbol{\omega}}_e(t) = -\tilde{\boldsymbol{\varphi}}_e(\mathbf{Me}, \mathbf{y}) \end{cases} \tag{7}$$

where the non-linear function $\tilde{\boldsymbol{\varphi}}_e(\mathbf{Me}, \mathbf{y})$ satisfies

$$\tilde{\boldsymbol{\varphi}}_e^T(\mathbf{Me}, \mathbf{y})\left[\tilde{\boldsymbol{\varphi}}_e(\mathbf{Me}, \mathbf{y}) - \tilde{\mathbf{K}}\mathbf{Me}(t)\right] \leq 0 \tag{8}$$

in which $\tilde{\mathbf{K}} = \mathbf{K}_2 - \mathbf{K}_1$.

Main results

Delay-dependent absolute stability of synchronization error

The following theorem shows the delay-dependent robust absolute stability of the error system (7).

Theorem 1. For the given controller gains $\mathbf{K}_e \in \mathbb{R}^{n \times n}$ and $\mathbf{K}_z \in \mathbb{R}^{n \times p}$, the non-linear delay system (7) with the non-linear function $\tilde{\boldsymbol{\varphi}}_e(\mathbf{Me}, \mathbf{y})$ satisfying (8) and $\tilde{\boldsymbol{\varphi}}(0, 0) = 0$ is absolutely stable if there exist scalar $\varepsilon > 0$, symmetric positive matrices $\mathbf{P}, \mathbf{Q}_j, \mathbf{C}_j, \mathbf{R}_j \in \mathbb{R}^{n \times n}$, $\mathbf{X}_j \in \mathbb{R}^{(2n+p)\times(2n+p)}$ and arbitrary matrices $\mathbf{Z}_{1j}$, $\mathbf{Z}_{2j} \in \mathbb{R}^{n \times n}$, and $\mathbf{Z}_{3j} \in \mathbb{R}^{p \times n}$, $(j = 1, \ldots, m+1)$ such that the following LMIs hold:

$$\boldsymbol{\Xi} = \begin{bmatrix} \boldsymbol{\eta} & \prod_1 & \prod_2 & \cdots & \prod_m & \boldsymbol{\theta}_1 & \boldsymbol{\Psi} & \mathbf{R}^1 & h(\tilde{\mathbf{A}} - \mathbf{K}_e)^T\mathbf{P} \\ * & \boldsymbol{\Sigma}_1 & 0 & \cdots & 0 & 0 & \boldsymbol{\Psi}_1 & & h\mathbf{A}_1^T\mathbf{P} \\ * & * & \boldsymbol{\Sigma}_2 & \cdots & 0 & 0 & \boldsymbol{\Psi}_2 & & h\mathbf{A}_2^T\mathbf{P} \\ * & * & * & \ddots & \vdots & \vdots & \vdots & \mathbf{R}^2 & \vdots \\ * & * & * & * & \boldsymbol{\Sigma}_m & 0 & \boldsymbol{\Psi}_m & & h\mathbf{A}_m^T\mathbf{P} \\ * & * & * & * & * & \boldsymbol{\theta}_2 & \boldsymbol{\theta}_3 & & h\mathbf{M}^T\mathbf{K}_z^T\mathbf{P} \\ * & * & * & * & * & * & \mho & 0 & h\mathbf{D}^T\mathbf{P} \\ * & * & * & * & * & * & * & \mathbf{C} & 0 \\ * & * & * & * & * & * & * & * & -h\mathbf{P} \end{bmatrix} < 0 \tag{9}$$

$$\prod_j = \begin{bmatrix} \mathbf{X}_{11}^j & \mathbf{X}_{12}^j & \mathbf{X}_{13}^j & \mathbf{Z}_{1j} \\ * & \mathbf{X}_{22}^j & \mathbf{X}_{23}^j & \mathbf{Z}_{2j} \\ * & * & \mathbf{X}_{33}^j & \mathbf{Z}_{3j} \\ * & * & * & \mathbf{P} \end{bmatrix} \geq 0, \ (j = 1, \ldots, m+1) \tag{10}$$

$$\mathbf{X}_j = \begin{bmatrix} \mathbf{X}_{11}^j & \mathbf{X}_{12}^j & \mathbf{X}_{13}^j \\ * & \mathbf{X}_{22}^j & \mathbf{X}_{23}^j \\ * & * & \mathbf{X}_{33}^j \end{bmatrix} > 0, \ (j = 1, \ldots, m+1) \tag{11}$$

where $\mathbf{X}_{11}^j, \mathbf{X}_{12}^j, \mathbf{X}_{22}^j \in \mathbb{R}^{n \times n}$, $\mathbf{X}_{13}^j, \mathbf{X}_{23}^j \in \mathbb{R}^{n \times p}$, $\mathbf{X}_{11}^j \in \mathbb{R}^{p \times p}$ *and*

$$\boldsymbol{\eta} = \tilde{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} - \mathbf{K}_e^T\mathbf{P} - \mathbf{P}\mathbf{K}_e + \sum_{j=1}^{m+1}\left(\mathbf{Q}_j + h_j\mathbf{C}_j + \mathbf{Z}_{1j} + \mathbf{Z}_{1j}^T + h_j\mathbf{X}_{11}^j\right)$$

$$\prod_j = \mathbf{P}\mathbf{A}_j - \mathbf{Z}_{1j} + \mathbf{Z}_{2j}^T + h_j\mathbf{X}_{12}^j$$

$$\boldsymbol{\Sigma}_j = -\mathbf{Q}_j - \mathbf{Z}_{2j} - \mathbf{Z}_{2j}^T + h_j\mathbf{X}_{22}^j, \ \boldsymbol{\Psi}_j = -\mathbf{Z}_{3j}^T + h_j\mathbf{X}_{23}^j$$

$$\boldsymbol{\theta}_3 = -\mathbf{Z}_{3(m+1)}^T + h_{m+1}\mathbf{X}_{23}^{m+1}$$

$$\boldsymbol{\Psi} = \mathbf{PD} - \varepsilon\mathbf{M}^T\tilde{\mathbf{K}}^T + \sum_{j=1}^{m+1}\left(\mathbf{Z}_{3j}^T + h_j\mathbf{X}_{13}^j\right)$$

$$\mho = -2\varepsilon\mathbf{I} + \sum_{j=1}^{m+1}\left(h_j\mathbf{X}_{33}^j\right)$$

$$\mathbf{C} = -\operatorname{diag}\{\mathbf{C}_1/h_1, \ldots, \mathbf{C}_{m+1}/h_{m+1}\}$$

$$h = \sum_{j=1}^{m+1}(h_j)$$

$$\boldsymbol{\theta}_1 = \mathbf{P}\mathbf{K}_z\mathbf{M} - \mathbf{Z}_{1(m+1)} + \mathbf{Z}_{2(m+1)}^T + h_{m+1}\mathbf{X}_{12}^{m+1}$$

$$\boldsymbol{\theta}_2 = -\mathbf{Q}_{m+1} - \mathbf{Z}_{2(m+1)} - \mathbf{Z}_{2(m+1)}^T + h_{m+1}\mathbf{X}_{22}^{m+1}$$

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{D}\mathbf{K}_1\mathbf{M}, \ \tilde{\mathbf{K}} = \mathbf{K}_2 - \mathbf{K}_1$$

$$\mathbf{R}^1 = [\mathbf{R}_1 \ \cdots \ \mathbf{R}_{m+1}], \ \mathbf{R}^2 = -\operatorname{diag}\{\mathbf{R}_1, \ldots, \mathbf{R}_{m+1}\}$$

Proof. Consider the following Lyapunov–Krasovskii functional:

$$V(\mathbf{e}_t) = V_1(\mathbf{e}_t) + V_2(\mathbf{e}_t) + V_3(\mathbf{e}_t) + V_4(\mathbf{e}_t) \tag{12}$$

where

$$V_1(\mathbf{e}_t) = \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t) + \sum_{j=1}^{m+1}\int_{t-h_j}^{t}\mathbf{e}^T(s)\mathbf{Q}_j\mathbf{e}(s)ds$$

$$V_2(\mathbf{e}_t) = \sum_{j=1}^{m+1}\left(\int_{t-h_j}^{t}\mathbf{e}^T(s)ds\right)\mathbf{R}_j\left(\int_{t-h_j}^{t}\mathbf{e}(s)ds\right)$$

$$V_3(\mathbf{e}_t) = \sum_{j=1}^{m+1}\int_{-h_j}^{0}\int_{t+\beta}^{t}\mathbf{e}^T(s)\mathbf{C}_j\mathbf{e}(s)dsd\beta$$

$$V_4(\mathbf{e}_t) = \sum_{j=1}^{m+1}\int_{-h_j}^{0}\int_{t+\beta}^{t}\dot{\mathbf{e}}^T(s)\mathbf{P}\,\dot{\mathbf{e}}(s)dsd\beta$$

in which $\mathbf{e}_t := \mathbf{e}(t+\theta)$ and $\theta \in [-\max_{1 \leq i \leq m+1}\{h_i\}, 0]$. Taking the derivative of $V_1(\mathbf{e}_t)$ and $V_2(\mathbf{e}_t)$ with respect to t yields

$$\begin{aligned} \dot{V}_1(\mathbf{e}_t) = {} & 2\mathbf{e}^T(t)\mathbf{P}\dot{\mathbf{e}}(t) + \mathbf{e}^T(t)\left(\sum_{j=1}^{m+1}\mathbf{Q}_j\right)\mathbf{e}(t) \\ & - \sum_{j=1}^{m+1}\mathbf{e}^T(t-h_j)\mathbf{Q}_j\mathbf{e}(t-h_j) \end{aligned} \tag{13}$$

$$\dot{V}_2(\mathbf{e}_t) = 2\sum_{j=1}^{m+1}\left(\mathbf{e}^T(t) - \mathbf{e}^T(t-h_j)\right)\mathbf{R}_j\left(\int_{t-h_j}^{t}\mathbf{e}(s)ds\right) \tag{14}$$

The third term of (12) becomes

$$\dot{V}_3(\mathbf{e}_t) = \mathbf{e}^T(t)(\sum_{j=1}^{m+1}h_j\mathbf{C}_j)\mathbf{e}(t) - \sum_{j=1}^{m+1}\int_{t-h_j}^{t}\mathbf{e}^T(s)\mathbf{C}_j\mathbf{e}(s)ds \tag{15}$$

Using the Jensen inequality (Gu and Niculescu, 2003), this gives

$$\dot{V}_3(\mathbf{e}_t) \leq \mathbf{e}^T(t)\left(\sum_{j=1}^{m+1} h_j\mathbf{C}_j\right)\mathbf{e}(t) - \sum_{j=1}^{m+1}\left(\int_{t-h_j}^{t}\mathbf{e}^T(s)ds\right)$$
$$\mathbf{C}_j/h_j\left(\int_{t-h_j}^{t}\mathbf{e}(s)ds\right) \tag{16}$$

Considering (12)–(22) and using the Schur complement (Boyd et al., 1994), it is straightforward to show that

$$\dot{V}(\mathbf{e}_t) \leq \boldsymbol{\xi}_1^T(t)\boldsymbol{\Xi}\ \boldsymbol{\xi}_1(t) - \sum_{j=1}^{m+1}\int_{t-h_j}^{t}\boldsymbol{\xi}_{2j}^T(t,s)\coprod_j\boldsymbol{\xi}_{2j}(t,s)ds \tag{23}$$

where $\boldsymbol{\Xi}$ and $\coprod_j$ are defined in (9) and (10), respectively, and

$$\boldsymbol{\xi}_1(t) = \begin{bmatrix}\mathbf{e}^T(t) & \mathbf{e}^T(t-h_1) & \cdots & \mathbf{e}^T(t-h_{m+1}) & \tilde{\boldsymbol{\omega}}_e^T(t) & \int_{t-h_1}^{t}\mathbf{e}^T(s)ds & \cdots & \int_{t-h_{m+1}}^{t}\mathbf{e}^T(s)ds\end{bmatrix}$$
$$\boldsymbol{\xi}_{2j}(t,s) = \begin{bmatrix}\mathbf{e}^T(t) & \mathbf{e}^T(t-h_j) & \tilde{\boldsymbol{\omega}}_e^T(t) & \dot{\mathbf{e}}^T(s)\end{bmatrix}^T$$

The fourth term of (12) becomes

$$\dot{V}_4(\mathbf{e}_t) = \dot{\mathbf{e}}^T(t)\left(\sum_{j=1}^{m+1} h_j\mathbf{P}\right)\dot{\mathbf{e}}(t) - \sum_{j=1}^{m+1}\int_{t-h_j}^{t}\dot{\mathbf{e}}^T(s)\mathbf{P}\,\dot{\mathbf{e}}(s)ds \tag{17}$$

The sector condition (8) can be written as

$$-\tilde{\boldsymbol{\omega}}_e^T(t)\left[\tilde{\boldsymbol{\omega}}_e(t) + \tilde{\mathbf{K}}\mathbf{M}\mathbf{e}(t)\right] \geq 0 \tag{18}$$

where $\tilde{\boldsymbol{\omega}}_e(t)$ is the same as in (7).

Noting that $\varepsilon > 0$, $\dot{V}(\mathbf{e}_t)$ can be expressed as

$$\begin{aligned}\dot{V}(\mathbf{e}_t) &\leq \dot{V}_1(\mathbf{e}_t) + \dot{V}_2(\mathbf{e}_t) + \dot{V}_3(\mathbf{e}_t) + \dot{V}_4(\mathbf{e}_t)\\ &\quad + \dot{V}_5(\mathbf{e}_t) - 2\varepsilon\tilde{\boldsymbol{\omega}}_e^T(t)\left[\tilde{\boldsymbol{\omega}}_e(t) + \tilde{\mathbf{K}}\mathbf{M}\mathbf{e}(t)\right]\end{aligned} \tag{19}$$

Using the Leibniz–Newton formula, this yields

$$\mathbf{e}(t) - \mathbf{e}(t-h_j) - \int_{t-h_j}^{t}\dot{\mathbf{e}}^T(s)ds = 0 \tag{20}$$

Then, for any constant matrices $\mathbf{Z}_{1j}$, $\mathbf{Z}_{2j}$, and $\mathbf{Z}_{3j}$ with appropriate dimensions, the following is true:

$$\begin{aligned}&\sum_{j=1}^{m+1}\left[\mathbf{e}^T(t)\mathbf{Z}_{1j} + \mathbf{e}^T(t-h_j)\mathbf{Z}_{2j} + \tilde{\boldsymbol{\omega}}_e^T(t)\mathbf{Z}_{3j}\right]\\ &\left[\mathbf{e}(t) - \mathbf{e}(t-h_j) - \int_{t-h_j}^{t}\dot{\mathbf{e}}^T(s)ds\right] = 0\end{aligned} \tag{21}$$

For any constant matrix $\mathbf{X}_j$ in (11) with appropriate dimensions, the following is true:

$$\begin{aligned}&\sum_{j=1}^{m+1}\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-h_j)\\ \tilde{\boldsymbol{\omega}}_e(t)\end{bmatrix}^T\\ &\begin{bmatrix}h_j(\mathbf{X}_{11}^j - \mathbf{X}_{11}^j) & h_j(\mathbf{X}_{12}^j - \mathbf{X}_{12}^j) & h_j(\mathbf{X}_{13}^j - \mathbf{X}_{13}^j)\\ * & h_j(\mathbf{X}_{22}^j - \mathbf{X}_{22}^j) & h_j(\mathbf{X}_{23}^j - \mathbf{X}_{23}^j)\\ * & * & h_j(\mathbf{X}_{33}^j - \mathbf{X}_{33}^j)\end{bmatrix}\\ &\begin{bmatrix}\mathbf{e}(t)\\ \mathbf{e}(t-h_j)\\ \tilde{\boldsymbol{\omega}}_e(t)\end{bmatrix} = 0\end{aligned} \tag{22}$$

If $\boldsymbol{\Xi} < 0$ and $\coprod_j \geq 0$, then $\dot{V}(\mathbf{e}_t) < 0$. □

Delay-dependent synchronization

In order to obtain the gain matrices $\mathbf{K}_e \in \mathbb{R}^{n\times n}$ and $\mathbf{K}_z \in \mathbb{R}^{n\times p}$ to synchronize the slave to the master, the following theorem can be used.

Theorem 2. The non-linear delay system (7) with the non-linear function $\tilde{\boldsymbol{\varphi}}_e(\mathbf{Me},\mathbf{y})$ satisfying (8) and $\tilde{\boldsymbol{\varphi}}(0,0) = 0$ is absolutely stable if there exist scalar $\varepsilon > 0$, symmetric positive matrices $\mathbf{P}$,$\mathbf{Q}_j$,$\mathbf{C}_j$,$\mathbf{R}_j \in \mathbb{R}^{n\times n}$, $\mathbf{X}_j \in \mathbb{R}^{(2n+p)\times(2n+p)}$, and arbitrary matrices $\mathbf{Y}_1 \in \mathbb{R}^{n\times n}$, $\mathbf{Y}_2 \in \mathbb{R}^{n\times p}$,$\mathbf{Z}_{1j}$, $\mathbf{Z}_{2j} \in \mathbb{R}^{n\times n}$ and $\mathbf{Z}_{3j} \in \mathbb{R}^{p\times n}$,$(j = 1, \ldots, m+1)$, such that the following LMIs hold:

$$\boldsymbol{\Xi} = \begin{bmatrix}\tilde{\boldsymbol{\eta}} & \coprod_1 & \coprod_2 & \cdots & \coprod_m & \tilde{\boldsymbol{\theta}}_1 & \boldsymbol{\Psi} & \mathbf{R}^1 & h\tilde{\mathbf{A}}^T\mathbf{P} - h\mathbf{Y}_1^T\\ * & \boldsymbol{\Sigma}_1 & 0 & \cdots & 0 & 0 & \boldsymbol{\Psi}_1 & & h\mathbf{A}_1^T\mathbf{P}\\ * & * & \boldsymbol{\Sigma}_2 & \cdots & 0 & 0 & \boldsymbol{\Psi}_2 & & h\mathbf{A}_2^T\mathbf{P}\\ * & * & * & \ddots & \vdots & \vdots & \vdots & \mathbf{R}^2 & \vdots\\ * & * & * & * & \boldsymbol{\Sigma}_m & 0 & \boldsymbol{\Psi}_m & & h\mathbf{A}_m^T\mathbf{P}\\ * & * & * & * & * & \boldsymbol{\theta}_2 & \boldsymbol{\theta}_3 & & h\mathbf{M}^T\mathbf{Y}_2^T\\ * & * & * & * & * & * & \mho & 0 & h\mathbf{D}^T\mathbf{P}\\ * & * & * & * & * & * & * & \mathbf{C} & 0\\ * & * & * & * & * & * & * & * & -h\mathbf{P}\end{bmatrix} < 0 \tag{24}$$

$$\coprod_j = \begin{bmatrix}\mathbf{X}_{11}^j & \mathbf{X}_{12}^j & \mathbf{X}_{13}^j & \mathbf{Z}_{1j}\\ * & \mathbf{X}_{22}^j & \mathbf{X}_{23}^j & \mathbf{Z}_{2j}\\ * & * & \mathbf{X}_{33}^j & \mathbf{Z}_{3j}\\ * & * & * & \mathbf{P}\end{bmatrix} \geq 0,\ (j = 1, \ldots, m+1) \tag{25}$$

$$\mathbf{X}_j = \begin{bmatrix}\mathbf{X}_{11}^j & \mathbf{X}_{12}^j & \mathbf{X}_{13}^j\\ * & \mathbf{X}_{22}^j & \mathbf{X}_{23}^j\\ * & * & \mathbf{X}_{33}^j\end{bmatrix} > 0,\ (j = 1, \ldots, m+1) \tag{26}$$

where $\mathbf{X}_{11}^j, \mathbf{X}_{12}^j, \mathbf{X}_{22}^j \in \mathbb{R}^{n\times n}, \mathbf{X}_{13}^j, \mathbf{X}_{23}^j \in \mathbb{R}^{n\times p}, \mathbf{X}_{11}^j \in \mathbb{R}^{p\times p}$, *and*

$$\tilde{\boldsymbol{\eta}} = \tilde{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} - \mathbf{Y}_1^T - \mathbf{Y}_1 + \sum_{j=1}^{m+1}\left(\mathbf{Q}_j + h_j\mathbf{C}_j + \mathbf{Z}_{1j} + \mathbf{Z}_{1j}^T + h_j\mathbf{X}_{11}^j\right)$$

$$\tilde{\boldsymbol{\theta}}_1 = \mathbf{Y}_2\mathbf{M} - \mathbf{Z}_{1(m+1)} + \mathbf{Z}_{2(m+1)}^T + h_{m+1}\mathbf{X}_{12}^{m+1}$$

Other parameters are the same as in (9). Moreover, the gain matrices are given by $\mathbf{K}_e = \mathbf{P}^{-1}\mathbf{Y}_1$ *and* $\mathbf{K}_z = \mathbf{P}^{-1}\mathbf{Y}_2$.

Proof. The proof is straightforward by substituting $\mathbf{Y}_1 = \mathbf{P}\mathbf{K}_e$ and $\mathbf{Y}_2 = \mathbf{P}\mathbf{K}_z$ in (9). □

Remark 3. In practice, the instantaneous states may not be available for the synchronization signal in (3), i.e. when $\mathbf{K}_e = \mathbf{0}$. In this case, $\mathbf{K}_e$ in Theorem 1 and $\mathbf{Y}_1$ in Theorem 2 should be ignored. In other words, $\mathbf{K}_e$ in Theorem 1 and $\mathbf{Y}_1$ in Theorem 2 should be replaced with zero matrices with the same dimensions.

Illustrative example

Consider the Chua's circuit with the following dimensionless equations (Chen et al., 2014):

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - m_1x_1(t) + f(x_1(t))) - 0.1x_1(t-h_1) \\ \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t) - 0.1x_1(t-h_1) \\ \dot{x}_3(t) = -bx_2(t) + 0.2x_1(t-h_1) - 0.1x_3(t-h_1) \end{cases} \tag{27}$$

where $f(x_1(t)) = 0.5(m_1 - m_0)(|x_1(t)+1| - |x_1(t)-1|)$ and a, b, m_0 and m_1 are constants. Assuming $a = 11$, $b = 14.286$, $m_0 = -1/7$ and $m_1 = 2/7$, a double scroll attractor will be obtained for $\mathbf{x}(0) = [-0.2 \quad -0.3 \quad 0.2]$.

System (27) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-h_1) + \mathbf{D}\omega(t) \\ z(t) = \mathbf{M}\mathbf{x}(t) \\ \omega(t) = -f(z(t)) \end{cases} \tag{28}$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \mathbf{A} = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}$$

$$\mathbf{A}_1 = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.1 & 0 & 0 \\ 0.2 & 0 & -0.1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix}$$

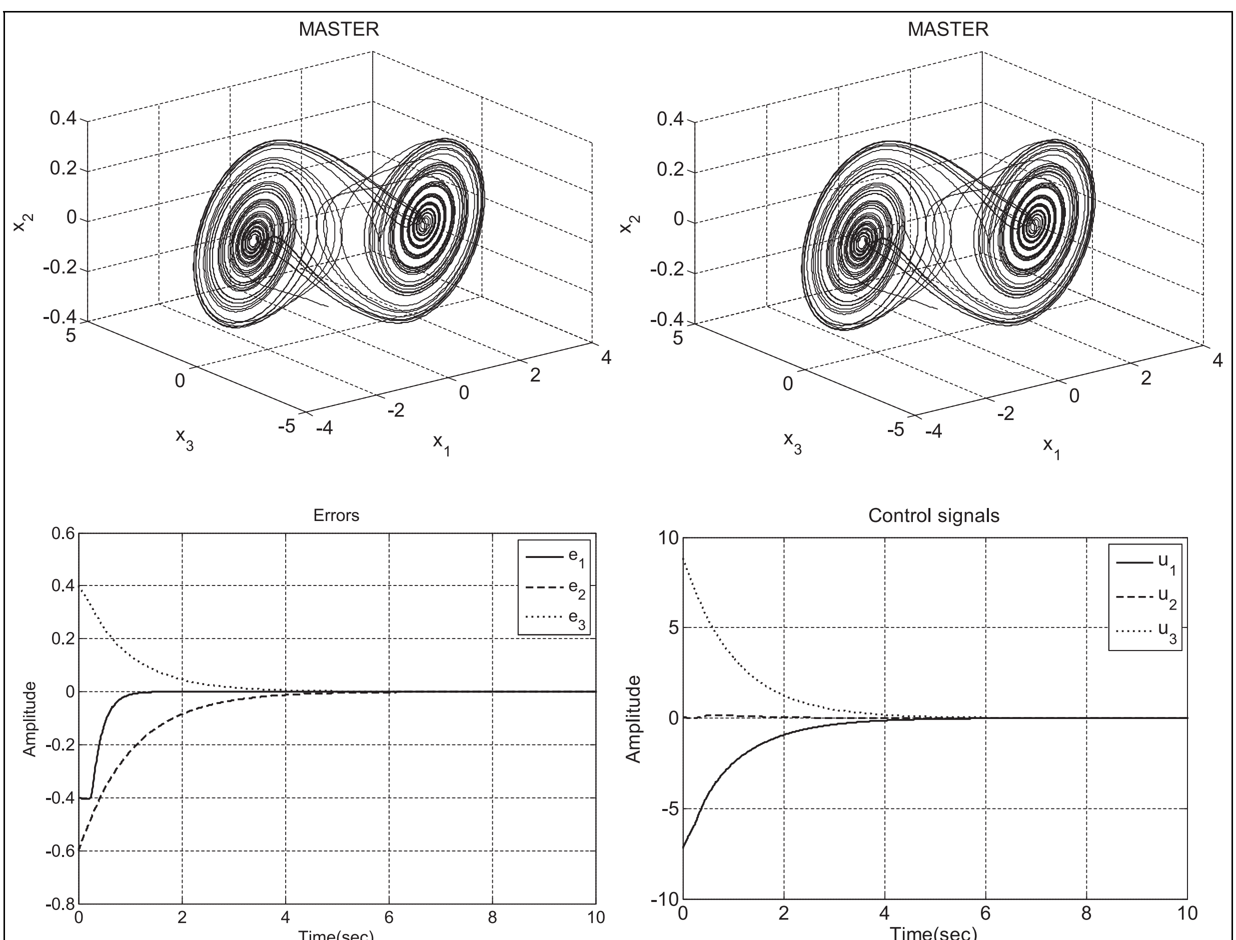

Figure 2. Simulation results for master, slave, errors, and control signals (Case 1).

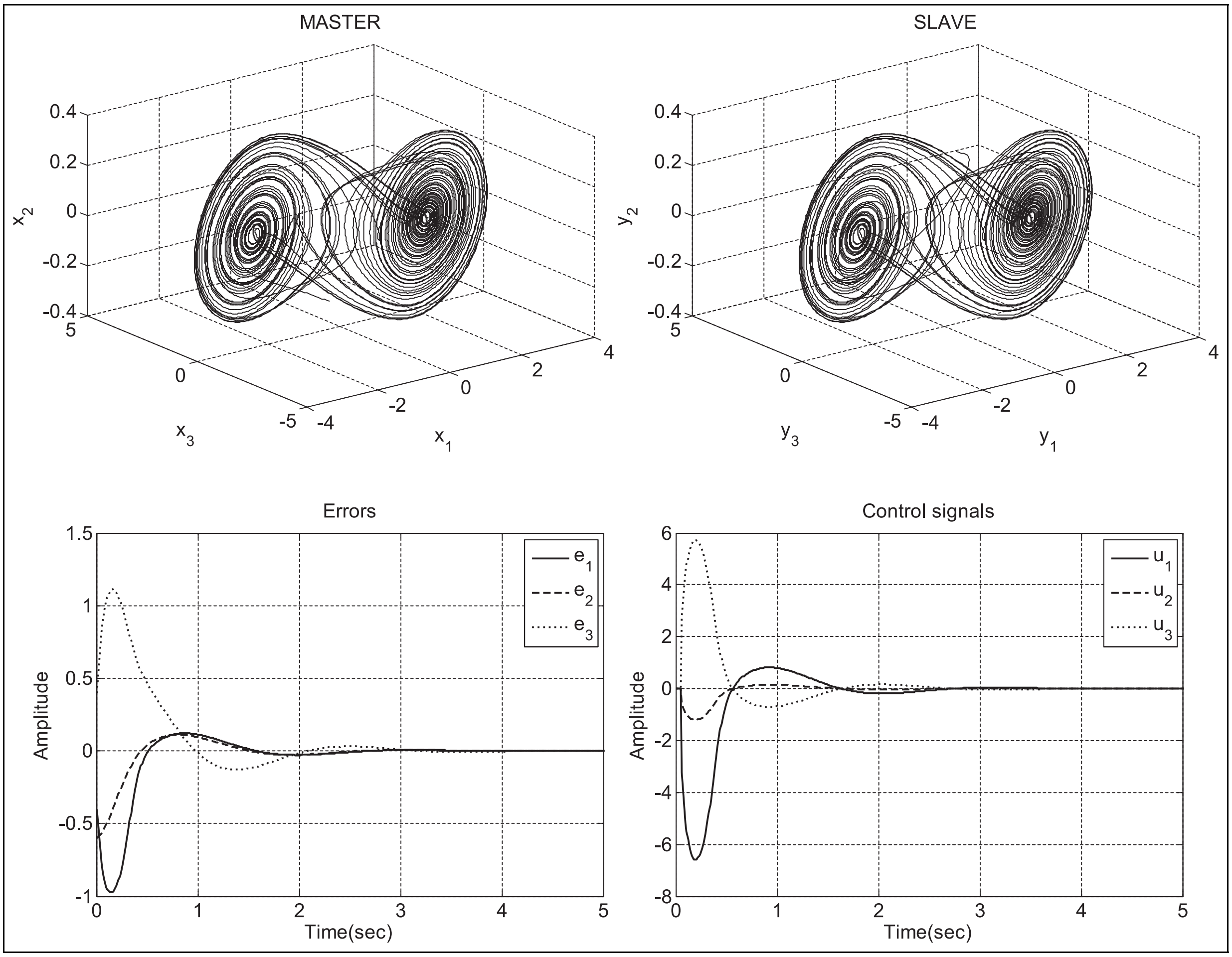

Figure 3. Simulation results for master, slave, errors, and control signals (Case 2).

$\mathbf{M} = [1 \quad 0 \quad 0], h_1 = 0.15$ and $f(z(t)) = 0.5(|z(t) + 1| - |z(t) - 1|)$. The non-linear function $f(z(t))$ in (27) is piecewise continuous in t, globally Lipschitz in $z(t)$, $f(0) = 0$ and belongs to sector $[0, k]$ with $k = 1$.

Case 1. Applying Theorem 2, the feedback gain matrices will be

$$\mathbf{K}_e = \begin{bmatrix} 2.7478 & 10.9922 & 0.0153 \\ 0.9389 & 0.0403 & 0.9851 \\ 0.1209 & -14.3008 & 1.0108 \end{bmatrix}, \mathbf{K}_z = \begin{bmatrix} -1.1109 \\ 0.0059 \\ -0.0118 \end{bmatrix}$$

The maximum allowed delay bound is $h_d = 0.2741$. Figure 2 shows the simulation results for the master, the slave, the errors and the control signals using these feedback gains. As this figure shows, the slave is synchronized with the master.

Case 2. When the instantaneous states in (3) are not available, then by applying Theorem 2 and assuming $\mathbf{K}_e = \mathbf{0}$, the feedback gain $\mathbf{K}_z$ will be

$$\mathbf{K}_z = \begin{bmatrix} -6.7800 \\ -1.2424 \\ 5.8752 \end{bmatrix}$$

The transmission delay bound will be $h_d = 0.05$. Figure 3 shows the simulation results for the master, the slave, the errors and the control signals using the above gains and $h_d = 0.05$.

Conclusion

This paper considered the problem of synchronization of the general Lur'e systems with multiple time delays using time-delay feedback control. Based on the Lyapunov–Krasovskii functional approach, some delay-dependent criteria were obtained and formulated in the form of the LMIs to guarantee global asymptotic stability of the error system such that the slave system is synchronized with the master system. The state feedback gains were obtained by solving these LMIs. The effectiveness of the proposed method was illustrated on the chaotic Chua's circuit.

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