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SYNCHRONIZATION CRITERIA FOR COMPLEX DYNAMICAL NETWORKS WITH STATE AND COUPLING TIME-DELAYS

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ABSTRACT

This paper is concerned with the problem of synchronization of complex dynamical networks with state and coupling time-delays. Therefore, larger class and more complicated complex dynamical networks could be considered for the synchronization problem. Based on the Lyapunov-Krasovskii functional, some delay-independent and delay-dependent criteria are obtained and formulated in the form of linear matrix inequalities (LMIs) to ascertain the synchronization between each node of the complex dynamical network. The effectiveness of the proposed method is illustrated using some numerical simulations.

Key Words: Complex dynamical network, synchronization, Lyapunov–Krasovskii, time-delayed systems.

I. INTRODUCTION

Many systems in the real-world can be modeled by networks, such as the neural networks, social network, electrical power grids, communication networks, the Internet, and the World Wide Web [1–3]. Complex networks are made up of interconnected nodes interacting with others via a topology defined on the network edges [4–7]. These nodes represent the individuals in the network with different meanings in different situations [8]. Each node of the network can be a nonlinear dynamical system and create a complex dynamical network (CDN), which has been widely applied to model many complex systems. In the past few decades, the study of CDNs has received increasing attention from researchers in various disciplines, such as physics, mathematics, engineering, biology and sociology [9–13].

Synchronization among all network's dynamical nodes is one of the most typical collective behaviors and basic motions in nature and is one of the most interesting and significant phenomena in CDNs [14–20]. In general, time delays occur commonly in networks because of the network traffic congestion as well as the finite speed of signal transmission over the links. Hence, the synchronization study of CDNs with coupling time delays is quite important [21–25]. Exponential synchronization in CDNs with time-varying delay and hybrid coupling is investigated in [23]. Guaranteed cost synchronization of CDNs is introduced in [26–28]. Complex dynamical networks with time-delay in the states of dynamical nodes have been rarely studied. In [29] and [30], synchronization criterion for Lur'e type complex dynamical networks is considered with time-delay in the states of the nodes and the coupling delay, respectively. To the best of the author's knowledge, almost all of the published papers have only considered the coupling delay for the network, but the state delay could exist in the nodes of the network. This is an area that has not yet been studied [28,31,32].

In this paper, synchronization criteria for CDNs with state and coupling time-delays are presented. Therefore, larger class and more complicated CDNs could be considered for the synchronization problem. Based on the Lyapunov-Krasovskii functional approach, some delay-independent and delay-dependent criteria are obtained and formulated in the form of LMIs. The effectiveness of the proposed method is illustrated using some numerical simulations.

The organization of this paper is as follows. In Section 2, the problem formulation for the complex dynamical network structure with state and coupling time-delays is presented. In Section 3, based on the Lyapunov–Krasovskii functional and LMI, some criteria are given to ascertain the synchronization between the nodes of CDNs. Section 4 provides simulation results. Finally, section 5 concludes the paper.

Notations. Throughout this paper, R^n denotes the n-dimensional Euclidean space and $R^{n\times m}$ is the set of real $n\times m$ matrices. $\mathbf{P}>0$ means that $\mathbf{P}$ is a real positive definite and symmetric matrix. $\mathbf{I}$ is the identity matrix with appropriate dimensions and diag$\{\mathbf{W}_1, \ldots, \mathbf{W}_m\}$ refers to a real matrix with diagonal elements $\mathbf{W}_1, \ldots, \mathbf{W}_m$. $\mathbf{A}^T$ denotes the transpose of the real matrix $\mathbf{A}$. Symmetric terms in a symmetric matrix are denoted by * and the sign ⊗ stands for the Kronecker product.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a complex dynamical network with N delayed identical nodes and coupling delay:

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \mathbf{A}\mathbf{x}_i(t) + \mathbf{A_d}\mathbf{x}_i(t-\tau) + \mathbf{B}\mathbf{f}(\mathbf{M}\mathbf{x}_i(t)) \\ &\quad + \mathbf{C}\mathbf{f}(\mathbf{D}\mathbf{x}_i(t-\tau)) + \sum_{j=1}^{N} G_{ij}^{(1)}\mathbf{\Gamma}_1\mathbf{x}_j(t) \\ &\quad + \sum_{j=1}^{N} G_{ij}^{(2)}\mathbf{\Gamma}_2\mathbf{x}_j(t-\tau_c),\ i=1,2,\ldots,N\end{aligned} \tag{1}$$

where $\mathbf{x}_i(t) = [x_{i1}(t) \quad x_{i2}(t) \quad \cdots \quad x_{in}(t)]^T \in R^n$ denotes the

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state vector of node $i, \mathbf{f}: R^n \rightarrow R^n$ is a nonlinear vector-valued function, $\mathbf{A}, \mathbf{A_d}, \mathbf{B}, \mathbf{M}, \mathbf{C}, \mathbf{D} \in R^{n\times n}$ are constant matrices, $\tau > 0$ denotes the state delay and $\tau_c > 0$ is the coupling delay. $\mathbf{G}^{(q)} = \left(G_{ij}^{(q)}\right)_{N\times N}$, $(q = 1, 2)$ denotes the coupling connections and $\mathbf{\Gamma}_1, \mathbf{\Gamma}_2 \in R^{n\times n}$ represents the inner coupling matrices.

Remark 1. In this research field, almost all of the related published papers only consider the coupling delay for the network, but the state delay could exist in the nodes of the network, and this has not yet been studied [28,31,32]. In this model, the state delay (τ) is considered for each node of the network, which is different from coupling delay (τ_c). In this way, more general complex dynamical networks could be modeled. To the best of the author's knowledge, a model with this configuration has not yet been studied

Assumption 1. The coupling connection matrices should satisfy:

$$\begin{cases} G_{ij}^{(q)} = G_{ji}^{(q)} \geq 0, & i \neq j,\ q = 1, 2, \\ G_{ii}^{(q)} = -\sum\limits_{j=1, j\neq i}^{N} G_{ij}^{(q)} \geq 0, & i, j = 1, \ldots, N,\ q = 1, 2. \end{cases}$$

Throughout this paper, I make the following assumption on $\mathbf{f}(\,.\,)$.

Assumption 2. For any $x_1, x_2 \in R$ there are some constants, σ_r^-, σ_r^+, which the nonlinear function satisfies:

$$\sigma_r^- \leq \frac{f_r(x_1) - f_r(x_2)}{x_1 - x_2} \leq \sigma_r^+, \qquad r = 1, 2, \ldots, n.$$

For notation simplicity, let

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} \mathbf{x}_1^T(t) & \mathbf{x}_2^T(t) & \cdots & \mathbf{x}_N^T(t) \end{bmatrix}^T \\ \mathbf{F}(\mathbf{x}(t)) &= \begin{bmatrix} \mathbf{f}^T(\mathbf{x}_1(t)) & \mathbf{f}^T(\mathbf{x}_2(t)) & \cdots & \mathbf{f}^T(\mathbf{x}_N(t)) \end{bmatrix}^T. \end{aligned}$$

With the help of the matrix Kronecker product, the network 1 can be written as the following form:

$$\begin{aligned} \dot{\mathbf{x}}(t) = {} & (\mathbf{I}_N \otimes \mathbf{A})\mathbf{x}(t) + (\mathbf{I}_N \otimes \mathbf{A_d})\mathbf{x}(t-\tau) \\ & + (\mathbf{I}_N \otimes \mathbf{B})\mathbf{F}((\mathbf{I}_N \otimes \mathbf{M})\mathbf{x}(t)) \\ & + (\mathbf{I}_N \otimes \mathbf{C})\mathbf{F}((\mathbf{I}_N \otimes \mathbf{D})\mathbf{x}(t-\tau)) + \left(\mathbf{G}^{(1)} \otimes \mathbf{\Gamma}_1\right)\mathbf{x}(t) \\ & + \left(\mathbf{G}^{(2)} \otimes \mathbf{\Gamma}_2\right)\mathbf{x}(t-\tau_c). \end{aligned} \tag{2}$$

The following definition and lemmas will be needed in the derivations of our main results.

Definition 1. The system 1 is said to be globally synchronized for any initial conditions: $\Pi_{i0}(s), (i = 1, 2, \ldots, N)$, if the following holds:

$$\lim_{t\rightarrow\infty} \left\| \mathbf{x}_i(t) - \mathbf{x}_j(t) \right\| = 0, \qquad \forall i, j = 1, 2, \ldots, N,$$

where$\|\,.\,\|$ denotes the Euclidean norm.

Lemma 1 ([10]). Let $\alpha \in R$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be matrices with appropriate dimensions. The following properties can be proved:

1. $(\alpha\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (\alpha\mathbf{B})$
2. $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$
3. $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD})$
4. $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C} = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$

Lemma 2. ((Jensen Inequality), [21]). Assume that the vector function $\boldsymbol{\omega}: [0, r] \rightarrow R^n$ is well defined for the following integrations. For any symmetric matrix $\mathbf{R} \in R^{n\times n}$ and scalar $r > 0$, one has:

$$r\int_0^r \boldsymbol{\omega}^T(s)\mathbf{R}\boldsymbol{\omega}(s)ds \geq \left(\int_0^r \boldsymbol{\omega}(s)ds\right)^T \mathbf{R}\left(\int_0^r \boldsymbol{\omega}(s)ds\right).$$

Lemma 3. According to [33] and Assumption 2, for any diagonal matrices $\mathbf{J}, \mathbf{L} > 0$, and constant matrix $\mathbf{M}$ with appropriate dimensions, it follows that:

$$\begin{aligned} & \boldsymbol{\theta}^T(t)\begin{bmatrix} -\mathbf{M}^T\mathbf{J}\mathbf{\Delta}_1\mathbf{M} & \mathbf{M}^T\mathbf{J}\mathbf{\Delta}_2 \\ * & -\mathbf{J} \end{bmatrix}\boldsymbol{\theta}(t) \\ & + \boldsymbol{\theta}^T(t-\tau)\begin{bmatrix} -\mathbf{M}^T\mathbf{L}\mathbf{\Delta}_1\mathbf{M} & \mathbf{M}^T\mathbf{L}\mathbf{\Delta}_2 \\ * & -\mathbf{L} \end{bmatrix}\boldsymbol{\theta}(t-\tau) \geq 0, \end{aligned} \tag{3}$$

where

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \mathbf{x}_i(t) - \mathbf{x}_j(t) \\ \mathbf{f}(\mathbf{M}\mathbf{x}_i(t)) - \mathbf{f}\left(\mathbf{M}\mathbf{x}_j(t)\right) \end{bmatrix},$$

$$\mathbf{\Delta}_1 = \operatorname{diag}\left[\sigma_1^+\sigma_1^-, \ldots, \sigma_n^+\sigma_n^-\right],$$

$$\mathbf{\Delta}_2 = \operatorname{diag}\left[\frac{\sigma_1^+ + \sigma_1^-}{2}, \ldots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right].$$

Lemma 4. ([31]). Let $\mathbf{e} = [1, 1, \ldots, 1]^T$, $\mathbf{E}_N = \mathbf{e}\mathbf{e}^T$, and $\mathbf{U} = N\mathbf{I}_N - \mathbf{E}_N$, $\mathbf{P} \in R^{n\times n}$, $\mathbf{x} = \left[\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T\right]^T$, and $\mathbf{y} = \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_N^T\right]^T$ with $\mathbf{x}_k, \mathbf{y}_k \in R^n$, $(k = 1, 2, \ldots, N)$, then

$$\mathbf{x}^T(\mathbf{U} \otimes \mathbf{P})\mathbf{y} = \sum_{1\leq i<j\leq N}^{N} \left(\mathbf{x}_i - \mathbf{x}_j\right)^T \mathbf{P}\left(\mathbf{y}_i - \mathbf{y}_j\right).$$

Lemma 5. ([31]). Let $\mathbf{H}$ and $\mathbf{S}$ be $n \times n$ any real matrix, $\mathbf{e} = [1, 1, \ldots, 1]^T$, $\mathbf{E}_N = \mathbf{e}\mathbf{e}^T$, $\mathbf{U} = N\mathbf{I}_N - \mathbf{E}_N$, $\mathbf{x} = \left[\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T\right]^T$, and $\mathbf{y} = \left[\mathbf{y}_1^T, \ldots, \mathbf{y}_N^T\right]^T$ with $\mathbf{x}_k, \mathbf{y}_k \in R^n$, $(k = 1, 2, \ldots, N)$, and $\mathbf{f}(\,.\,)$,$\mathbf{F}(\,.\,)$ are functions and defined in 2. Then, for any vectors $\mathbf{x}$ and $\mathbf{y}$ with appropriate dimensions, the following matrix inequality holds:

$$\mathbf{x}^T(\mathbf{U} \otimes \mathbf{H})\mathbf{F}((I_N \otimes \mathbf{S})\mathbf{y}) = \sum_{1\leq i<j\leq N}^{N} \left(\mathbf{x}_i - \mathbf{x}_j\right)^T \mathbf{H}\left(\mathbf{f}(\mathbf{S}\mathbf{y}_i) - \mathbf{f}\left(\mathbf{S}\mathbf{y}_j\right)\right).$$

III. MAIN RESULTS

In this chapter, some sufficient conditions based on the Lyapunov-Krasovskii functional method will be presented for the synchronization between the nodes of complex dynamical network 2.

Theorem 1. The system 2 is globally synchronized if there exist positive definite matrices $\mathbf{P}, \mathbf{R} \in R^{n\times n} > 0$, $\mathbf{Q} \in R^{3n\times 3n} > 0$, and positive diagonal matrices $\mathbf{J}_1, \mathbf{J}_2, \mathbf{L}_1, \mathbf{L}_2 \in R^{n\times n} > 0$, such that the following LMIs hold for all $1 \le i < j \le N$:

$$\boldsymbol{\Xi}_{ij} = \begin{bmatrix} \Pi_{11} & \mathbf{PA}_d & \Pi_{13} & \Pi_{14} & 0 & \Pi_{16} & \mathbf{PC} \\ * & \Pi_{22} & 0 & 0 & \Pi_{25} & 0 & \Pi_{27} \\ * & * & -\mathbf{R} & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & 0 & \mathbf{Q}_{23} & 0 \\ * & * & * & * & \Pi_{55} & 0 & -\mathbf{Q}_{23} \\ * & * & * & * & * & \Pi_{66} & 0 \\ * & * & * & * & * & * & \Pi_{77} \end{bmatrix} < 0, \tag{4}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ * & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ * & * & \mathbf{Q}_{33} \end{bmatrix} > 0, \tag{5}$$

where

$$\begin{aligned}
\Pi_{11} &= \mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{Q}_{11} + \mathbf{R} - \mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_1\mathbf{M} - \mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_1\mathbf{D} \\
&\quad -NG_{ij}^{(1)}\mathbf{P}\boldsymbol{\Gamma}_1 - NG_{ij}^{(1)}\boldsymbol{\Gamma}_1^T\mathbf{P}, \Pi_{13} = -NG_{ij}^{(2)}\mathbf{P}\boldsymbol{\Gamma}_2, \\
\Pi_{14} &= \mathbf{PB} + \mathbf{Q}_{12} + \mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_2, \Pi_{16} = \mathbf{Q}_{13} + \mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_2, \\
\Pi_{22} &= -\mathbf{Q}_{11} - \mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_1\mathbf{M} - \mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_1\mathbf{D}, \\
\Pi_{25} &= -\mathbf{Q}_{12} + \mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_2, \Pi_{27} = -\mathbf{Q}_{13} + \mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_2, \\
\Pi_{44} &= \mathbf{Q}_{22} - \mathbf{J}_1, \Pi_{55} = -\mathbf{Q}_{22} - \mathbf{L}_1, \Pi_{66} = \mathbf{Q}_{33} - \mathbf{J}_2, \\
\Pi_{77} &= -\mathbf{Q}_{33} - \mathbf{L}_2.
\end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{6}$$

where

$$\begin{aligned}
V_1(t) &= \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{P})\mathbf{x}(t), \\
V_2(t) &= \int_{t-\tau}^{t} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{F}(\mathbf{Mx}(s)) \\ \mathbf{F}(\mathbf{Dx}(s)) \end{bmatrix}^T \begin{bmatrix} \mathbf{U}\otimes\mathbf{Q}_{11} & \mathbf{U}\otimes\mathbf{Q}_{12} & \mathbf{U}\otimes\mathbf{Q}_{13} \\ * & \mathbf{U}\otimes\mathbf{Q}_{22} & \mathbf{U}\otimes\mathbf{Q}_{23} \\ * & * & \mathbf{U}\otimes\mathbf{Q}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{F}(\mathbf{Mx}(s)) \\ \mathbf{F}(\mathbf{Dx}(s)) \end{bmatrix} ds, \\
V_3(t) &= \int_{t-\tau_c}^{t} \mathbf{x}^T(s)(\mathbf{U}\otimes\mathbf{R})\mathbf{x}(s)ds,
\end{aligned}$$

where $\mathbf{U}$ is defined in Lemma 4.

Taking the derivative of $V_1(t)$ with respect to t yields:

$$\begin{aligned}
\dot{V}_1(t) &= 2\mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{P})\dot{\mathbf{x}}(t) = 2\mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{P})[(\mathbf{I}_N\otimes\mathbf{A})\mathbf{x}(t) \\
&\quad +(\mathbf{I}_N\otimes\mathbf{A_d})\mathbf{x}(t-\tau) + (\mathbf{I}_N\otimes\mathbf{B})\mathbf{F}((\mathbf{I}_N\otimes\mathbf{M})\mathbf{x}(t)) \\
&\quad +(\mathbf{I}_N\otimes\mathbf{C})\mathbf{F}((\mathbf{I}_N\otimes\mathbf{D})\mathbf{x}(t-\tau)) + \left(\mathbf{G}^{(1)}\otimes\boldsymbol{\Gamma}_1\right)\mathbf{x}(t) \\
&\quad +\left(\mathbf{G}^{(2)}\otimes\boldsymbol{\Gamma}_2\right)\mathbf{x}(t-\tau_c)]
\end{aligned} \tag{7}$$

According to Lemma 4, 7 can be written as the following:

$$\begin{aligned}
\dot{V}_1(t) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} &[2\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right)^T[\mathbf{PA}\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right) \\
&+\mathbf{PA}_d\left(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)\right) \\
&+\mathbf{PB}\left(\mathbf{f}(\mathbf{Mx}_i(t)) - \mathbf{f}\left(\mathbf{Mx}_j(t)\right)\right) \\
&+\mathbf{PC}\left(\mathbf{f}(\mathbf{Dx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Dx}_j(t-\tau)\right)\right) \\
&-NG_{ij}^{(1)}\mathbf{P}\boldsymbol{\Gamma}_1\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right) \\
&-NG_{ij}^{(2)}\mathbf{P}\boldsymbol{\Gamma}_2\left(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c)\right)]].
\end{aligned} \tag{8}$$

The second term of 6 becomes

$$\begin{aligned}
\dot{V}_2(t) = &\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{F}(\mathbf{Mx}(t)) \\ \mathbf{F}(\mathbf{Dx}(t)) \end{bmatrix}^T \begin{bmatrix} \mathbf{U}\otimes\mathbf{Q}_{11} & \mathbf{U}\otimes\mathbf{Q}_{12} & \mathbf{U}\otimes\mathbf{Q}_{13} \\ * & \mathbf{U}\otimes\mathbf{Q}_{22} & \mathbf{U}\otimes\mathbf{Q}_{23} \\ * & * & \mathbf{U}\otimes\mathbf{Q}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{F}(\mathbf{Mx}(t)) \\ \mathbf{F}(\mathbf{Dx}(t)) \end{bmatrix} \\
&- \begin{bmatrix} \mathbf{x}(t-\tau) \\ \mathbf{F}(\mathbf{Mx}(t-\tau)) \\ \mathbf{F}(\mathbf{Dx}(t-\tau)) \end{bmatrix}^T \begin{bmatrix} \mathbf{U}\otimes\mathbf{Q}_{11} & \mathbf{U}\otimes\mathbf{Q}_{12} & \mathbf{U}\otimes\mathbf{Q}_{13} \\ * & \mathbf{U}\otimes\mathbf{Q}_{22} & \mathbf{U}\otimes\mathbf{Q}_{23} \\ * & * & \mathbf{U}\otimes\mathbf{Q}_{33} \end{bmatrix} \\
&\begin{bmatrix} \mathbf{x}(t-\tau) \\ \mathbf{F}(\mathbf{Mx}(t-\tau)) \\ \mathbf{F}(\mathbf{Dx}(t-\tau)) \end{bmatrix}
\end{aligned} \tag{9}$$

According to Lemma 4 and 5, 9 can be written as the following:

$$\begin{aligned}
\dot{V}_2(t) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} &\Big[\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right)^T\mathbf{Q}_{11}\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right) \\
&+ 2\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right)^T\mathbf{Q}_{12}\left(\mathbf{f}(\mathbf{Mx}_i(t)) - \mathbf{f}\left(\mathbf{Mx}_j(t)\right)\right) \\
&+2\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right)^T\mathbf{Q}_{13}\left(\mathbf{f}(\mathbf{Dx}_i(t)) - \mathbf{f}\left(\mathbf{Dx}_j(t)\right)\right) \\
&+\left(\mathbf{f}(\mathbf{Mx}_i(t)) - \mathbf{f}\left(\mathbf{Mx}_j(t)\right)\right)^T\mathbf{Q}_{22}\left(\mathbf{f}(\mathbf{Mx}_i(t)) - \mathbf{f}\left(\mathbf{Mx}_j(t)\right)\right) \\
&+2\left(\mathbf{f}(\mathbf{Mx}_i(t)) - \mathbf{f}\left(\mathbf{Mx}_j(t)\right)\right)^T\mathbf{Q}_{23}\left(\mathbf{f}(\mathbf{Dx}_i(t)) - \mathbf{f}\left(\mathbf{Dx}_j(t)\right)\right) \\
&+\left(\mathbf{f}(\mathbf{Dx}_i(t)) - \mathbf{f}\left(\mathbf{Dx}_j(t)\right)\right)^T\mathbf{Q}_{33}\left(\mathbf{f}(\mathbf{Dx}_i(t)) - \mathbf{f}\left(\mathbf{Dx}_j(t)\right)\right) \\
&-\left(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)\right)^T\mathbf{Q}_{11}\left(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)\right) \\
&-2\left(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)\right)^T\mathbf{Q}_{12}\left(\mathbf{f}(\mathbf{Mx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Mx}_j(t-\tau)\right)\right) \\
&-2\left(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)\right)^T\mathbf{Q}_{13}\left(\mathbf{f}(\mathbf{Dx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Dx}_j(t-\tau)\right)\right) \\
&-\left(\mathbf{f}(\mathbf{Mx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Mx}_j(t-\tau)\right)\right)^T\mathbf{Q}_{22}\left(\mathbf{f}(\mathbf{Mx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Mx}_j(t-\tau)\right)\right) \\
&-2\left(\mathbf{f}(\mathbf{Mx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Mx}_j(t-\tau)\right)\right)^T\mathbf{Q}_{23}\left(\mathbf{f}(\mathbf{Dx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Dx}_j(t-\tau)\right)\right) \\
&-\left(\mathbf{f}(\mathbf{Dx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Dx}_j(t-\tau)\right)\right)^T\mathbf{Q}_{33}\left(\mathbf{f}(\mathbf{Dx}_i(t-\tau)) - \mathbf{f}\left(\mathbf{Dx}_j(t-\tau)\right)\right)\Big]
\end{aligned} \tag{10}$$

The third term of 6 becomes

$$\begin{aligned}
\dot{V}_3(t) &= \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{R})\mathbf{x}(t) - \mathbf{x}^T(t-\tau_c)(\mathbf{U}\otimes\mathbf{R})\mathbf{x}(t-\tau_c) \\
&= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} [\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right)^T\mathbf{R}\left(\mathbf{x}_i(t) - \mathbf{x}_j(t)\right) \\
&\quad -\left(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c)\right)^T\mathbf{R}\left(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c)\right)]
\end{aligned} \tag{11}$$

According to Lemma 3 and Assumption 2, for any positive diagonal matrices $\mathbf{J}_1, \mathbf{J}_2, \mathbf{L}_1, \mathbf{L}_2 \in R^{n\times n} > 0$, one has

$$\begin{aligned}
&\boldsymbol{\theta}^T(t)\begin{bmatrix} -\mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_1\mathbf{M} & \mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_2 \\ * & -\mathbf{J}_1 \end{bmatrix}\boldsymbol{\theta}(t) \\
&\quad +\boldsymbol{\theta}^T(t-\tau)\begin{bmatrix} -\mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_1\mathbf{M} & \mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_2 \\ * & -\mathbf{L}_1 \end{bmatrix}\boldsymbol{\theta}(t-\tau) \ge 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\boldsymbol{\beta}^T(t)\begin{bmatrix} -\mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_1\mathbf{D} & \mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_2 \\ * & -\mathbf{J}_2 \end{bmatrix}\boldsymbol{\beta}(t) \\
&\quad + \boldsymbol{\beta}^T(t-\tau)\begin{bmatrix} -\mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_1\mathbf{D} & \mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_2 \\ * & -\mathbf{L}_2 \end{bmatrix}\boldsymbol{\beta}(t-\tau) \ge 0,
\end{aligned} \tag{13}$$

where

$$\boldsymbol{\theta}(t) = \begin{bmatrix} \mathbf{x}_i(t) - \mathbf{x}_j(t) \\ \mathbf{f}(\mathbf{M}\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{M}\mathbf{x}_j(t)) \end{bmatrix},$$

$$\boldsymbol{\beta}(t) = \begin{bmatrix} \mathbf{x}_i(t) - \mathbf{x}_j(t) \\ \mathbf{f}(\mathbf{D}\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{D}\mathbf{x}_j(t)) \end{bmatrix}.$$

Considering 7–13, it is straightforward to show that

$$\dot{V}(t) \leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left[\boldsymbol{\xi}_{ij}^T(t) \boldsymbol{\Xi}_{ij} \boldsymbol{\xi}_{ij}(t) \right] \tag{14}$$

where $\boldsymbol{\Xi}_{ij}$ is defined in 4 and

$$\begin{aligned} \boldsymbol{\xi}_{ij}(t) = [\, &(\mathbf{x}_i(t) - \mathbf{x}_j(t))^T \quad (\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau))^T \\ &(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c))^T (\mathbf{f}(\mathbf{M}\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{M}\mathbf{x}_j(t)))^T \ldots \\ &\cdots \ (\mathbf{f}(\mathbf{M}\mathbf{x}_i(t-\tau)) - \mathbf{f}(\mathbf{M}\mathbf{x}_j(t-\tau)))^T \\ &(\mathbf{f}(\mathbf{D}\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{D}\mathbf{x}_j(t)))^T (\mathbf{f}(\mathbf{D}\mathbf{x}_i(t-\tau)) - \mathbf{f}(\mathbf{D}\mathbf{x}_j(t-\tau)))^T]^T. \end{aligned}$$

If $\boldsymbol{\Xi}_{ij} < 0$ for $\forall\, 1 \leq i < j \leq N$, then $\dot{V}(t) < 0$. From Definition 1, this implies that the system 2 has a global synchronization. This completes the proof. □

Remark 2. Theorem 1 provides delay-independent criterion. If Theorem 1 were satisfied for a system, then the system would have global synchronization for any state delay (τ) and coupling delay (τ_c). Obviously this criterion is a very conservative condition.

The next theorem provides a delay-dependent criterion by considering a new Lyapunov-Krasovskii functional. This theorem is less conservative in comparison with Theorem 1.

Theorem 2. The system 2 is globally synchronized if there exist positive definite matrices $\mathbf{P}, \mathbf{R}, \mathbf{Z}, \mathbf{O} \in R^{n \times n} > 0$, $\mathbf{Q}$, $\mathbf{W} \in R^{3n \times 3n} > 0$, and positive diagonal matrices $\mathbf{J}_1, \mathbf{J}_2, \mathbf{L}_1$, $\mathbf{L}_2 \in R^{n \times n} > 0$, such that the following LMIs hold for all $1 \leq i < j \leq N$:

$$\boldsymbol{\Psi}_{ij} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & 0 & \Pi_{16} & \Pi_{17} & \Pi_{18} & \Pi_{19} \\ * & \Pi_{22} & 0 & 0 & \Pi_{25} & 0 & \Pi_{27} & \Pi_{28} & \Pi_{29} \\ * & * & -\mathbf{R} & 0 & 0 & 0 & 0 & \Pi_{38} & \Pi_{39} \\ * & * & * & \Pi_{44} & 0 & \mathbf{Q}_{23} & 0 & \mathbf{B}^T\mathbf{W}_{12} & \mathbf{B}^T\mathbf{W}_{13} \\ * & * & * & * & \Pi_{55} & 0 & -\mathbf{Q}_{23} & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Pi_{77} & \mathbf{C}^T\mathbf{W}_{12} & \mathbf{C}^T\mathbf{W}_{13} \\ * & * & * & * & * & * & * & \frac{1}{\tau}\mathbf{Z} & 0 \\ * & * & * & * & * & * & * & * & \frac{1}{\tau_c}\mathbf{O} \end{bmatrix} < 0, \tag{15}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ * & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ * & * & \mathbf{Q}_{33} \end{bmatrix} > 0, \tag{16}$$

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} & \mathbf{W}_{13} \\ * & \mathbf{W}_{22} & \mathbf{W}_{23} \\ * & * & \mathbf{W}_{33} \end{bmatrix} > 0$$

where

$$\begin{aligned} \Pi_{11} = {} & \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - NG_{ij}^{(1)}(\mathbf{P}\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_1^T\mathbf{P}) + \mathbf{Q}_{11} + \mathbf{R} \\ & + \mathbf{W}_{11}\mathbf{A} + \mathbf{A}^T\mathbf{W}_{11} - NG_{ij}^{(1)}(\mathbf{W}_{11}\boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_1^T\mathbf{W}_{11}) \\ & + \mathbf{W}_{12} + \mathbf{W}_{12}^T + \mathbf{W}_{13} + \mathbf{W}_{13}^T + \tau\mathbf{Z} + \tau_c\mathbf{O} \\ & - \mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_1\mathbf{M} - \mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_1\mathbf{D}, \Pi_{12} = \mathbf{P}\mathbf{A}_d + \mathbf{W}_{11}\mathbf{A}_d - \mathbf{W}_{12}, \\ \Pi_{13} = {} & -NG_{ij}^{(2)}(\mathbf{P} + \mathbf{W}_{11})\boldsymbol{\Gamma}_2 - \mathbf{W}_{13}, \Pi_{14} = \mathbf{P}\mathbf{B} + \mathbf{Q}_{12} \\ & + \mathbf{W}_{11}\mathbf{B} + \mathbf{M}^T\mathbf{J}_1\boldsymbol{\Delta}_2, \Pi_{16} = \mathbf{Q}_{13} + \mathbf{D}^T\mathbf{J}_2\boldsymbol{\Delta}_2, \\ \Pi_{17} = {} & \mathbf{P}\mathbf{C} + \mathbf{W}_{11}\mathbf{C}, \Pi_{18} = \mathbf{A}^T\mathbf{W}_{12} - NG_{ij}^{(1)}\boldsymbol{\Gamma}_1^T\mathbf{W}_{12} \\ & + \mathbf{W}_{22} + \mathbf{W}_{23}^T, \Pi_{19} = \mathbf{A}^T\mathbf{W}_{13} - NG_{ij}^{(1)}\boldsymbol{\Gamma}_1^T\mathbf{W}_{13} \\ & + \mathbf{W}_{23} + \mathbf{W}_{33}, \Pi_{22} = -\mathbf{Q}_{11} - \mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_1\mathbf{M} \\ & - \mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_1\mathbf{D}, \Pi_{25} = -\mathbf{Q}_{12} + \mathbf{M}^T\mathbf{L}_1\boldsymbol{\Delta}_2, \\ \Pi 27 = {} & -\mathbf{Q}_{13} + \mathbf{D}^T\mathbf{L}_2\boldsymbol{\Delta}_2, \Pi_{28} = \mathbf{A}_d^T\mathbf{W}_{12} - \mathbf{W}_{22}, \\ \Pi_{29} = {} & \mathbf{A}_d^T\mathbf{W}_{13} - \mathbf{W}_{23}, \Pi_{38} = -NG_{ij}^{(2)}\boldsymbol{\Gamma}_2^T\mathbf{W}_{12} - \mathbf{W}_{23}^T, \\ \Pi 39 = {} & -NG_{ij}^{(2)}\boldsymbol{\Gamma}_2^T\mathbf{W}_{13} - \mathbf{W}_{33}, \Pi_{44} = \mathbf{Q}_{22} - \mathbf{J}_1, \\ \Pi_{55} = {} & -\mathbf{Q}_{22} - \mathbf{L}_1, \Pi_{66} = \mathbf{Q}_{33} - \mathbf{J}_2, \Pi_{77} = -\mathbf{Q}_{33} - \mathbf{L}_2. \end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t), \tag{17}$$

where $V_1(t)$, $V_2(t)$, and $V_3(t)$ are defined in 6 and

$$V_4(t) = \begin{bmatrix} \mathbf{x}(t) \\ \int_{t-\tau}^t \mathbf{x}(s)ds \\ \int_{t-\tau_c}^t \mathbf{x}(s)ds \end{bmatrix}^T \begin{bmatrix} \mathbf{U}\otimes\mathbf{W}_{11} & \mathbf{U}\otimes\mathbf{W}_{12} & \mathbf{U}\otimes\mathbf{W}_{13} \\ * & \mathbf{U}\otimes\mathbf{W}_{22} & \mathbf{U}\otimes\mathbf{W}_{23} \\ * & * & \mathbf{U}\otimes\mathbf{W}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \int_{t-\tau}^t \mathbf{x}(s)ds \\ \int_{t-\tau_c}^t \mathbf{x}(s)ds \end{bmatrix},$$

$$V_5(t) = \int_{t-\tau}^t \int_\theta^t \mathbf{x}^T(s)(\mathbf{U}\otimes\mathbf{Z})\mathbf{x}(s)dsd\theta + \int_{t-\tau_c}^t \int_\theta^t \mathbf{x}^T(s)(\mathbf{U}\otimes\mathbf{O})\mathbf{x}(s)dsd\theta.$$

where $\mathbf{U}$ is defined in Lemma 4.

The derivatives of $V_1(t)$, $V_2(t)$, and $V_3(t)$ are presented in 8–11. Taking the derivative of $V_4(t)$ with respect to t yields:

$$\dot{V}_4(t) = 2 \begin{bmatrix} \mathbf{x}(t) \\ \int_{t-\tau}^t \mathbf{x}(s)ds \\ \int_{t-\tau_c}^t \mathbf{x}(s)ds \end{bmatrix}^T \begin{bmatrix} \mathbf{U}\otimes\mathbf{W}_{11} & \mathbf{U}\otimes\mathbf{W}_{12} & \mathbf{U}\otimes\mathbf{W}_{13} \\ * & \mathbf{U}\otimes\mathbf{W}_{22} & \mathbf{U}\otimes\mathbf{W}_{23} \\ * & * & \mathbf{U}\otimes\mathbf{W}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{x}(t) - \mathbf{x}(t-\tau) \\ \mathbf{x}(t) - \mathbf{x}(t-\tau_c) \end{bmatrix} \tag{18}$$

According to Lemma 4, 18 can be written as the following:

$$\begin{aligned} \dot{V}_4(t) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \Big[& 2(\mathbf{x}_i(t) - \mathbf{x}_j(t))^T (\mathbf{W}_{11}(\dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_j(t)) + (\mathbf{W}_{12} + \mathbf{W}_{13})(\mathbf{x}_i(t) - \mathbf{x}_j(t))) \\ & - 2(\mathbf{x}_i(t) - \mathbf{x}_j(t))^T (\mathbf{W}_{12}(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)) + \mathbf{W}_{13}(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c))) \\ & + 2\left(\int_{t-\tau}^t \mathbf{x}_i(s)ds - \int_{t-\tau}^t \mathbf{x}_j(s)ds\right)^T (\mathbf{W}_{12}^T(\dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_j(t)) + (\mathbf{W}_{22} + \mathbf{W}_{23})(\mathbf{x}_i(t) - \mathbf{x}_j(t))) \\ & - 2\left(\int_{t-\tau}^t \mathbf{x}_i(s)ds - \int_{t-\tau}^t \mathbf{x}_j(s)ds\right)^T \mathbf{W}_{22}(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)) \\ & - 2\left(\int_{t-\tau}^t \mathbf{x}_i(s)ds - \int_{t-\tau}^t \mathbf{x}_j(s)ds\right)^T \mathbf{W}_{23}(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c)) \\ & + 2\left(\int_{t-\tau_c}^t \mathbf{x}_i(s)ds - \int_{t-\tau_c}^t \mathbf{x}_j(s)ds\right)^T (\mathbf{W}_{13}^T(\dot{\mathbf{x}}_i(t) - \dot{\mathbf{x}}_j(t)) + (\mathbf{W}_{23}^T + \mathbf{W}_{33})(\mathbf{x}_i(t) - \mathbf{x}_j(t))) \\ & - 2\left(\int_{t-\tau_c}^t \mathbf{x}_i(s)ds - \int_{t-\tau_c}^t \mathbf{x}_j(s)ds\right)^T \mathbf{W}_{23}^T(\mathbf{x}_i(t-\tau) - \mathbf{x}_j(t-\tau)) \\ & - 2\left(\int_{t-\tau_c}^t \mathbf{x}_i(s)ds - \int_{t-\tau_c}^t \mathbf{x}_j(s)ds\right)^T \mathbf{W}_{33}(\mathbf{x}_i(t-\tau_c) - \mathbf{x}_j(t-\tau_c)) \Big]. \end{aligned} \tag{19}$$

Taking the derivative of $V_5(t)$ with respect to t yields:

$$
\begin{aligned}
\dot{V}_5(t) &= \tau \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{Z})\mathbf{x}(t) \\
&\quad -\int_{t-\tau}^{t} \mathbf{x}^T(s)(\mathbf{U}\otimes\mathbf{Z})\mathbf{x}(s)ds \\
&\quad +\tau_c \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{O})\mathbf{x}(t) \\
&\quad -\int_{t-\tau_c}^{t} \mathbf{x}^T(s)(\mathbf{U}\otimes\mathbf{O})\mathbf{x}(s)ds.
\end{aligned} \tag{20}
$$

According to Lemma 2, 20 can be written as the following:

$$
\begin{aligned}
\dot{V}_5(t) &\le \tau \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{Z})\mathbf{x}(t) - \frac{1}{\tau}\left(\int_{t-\tau}^{t}\mathbf{x}(s)ds\right)^T(\mathbf{U}\otimes\mathbf{Z})\left(\int_{t-\tau}^{t}\mathbf{x}(s)ds\right) \\
&\quad + \tau_c \mathbf{x}^T(t)(\mathbf{U}\otimes\mathbf{O})\mathbf{x}(t) - \frac{1}{\tau_c}\left(\int_{t-\tau_c}^{t}\mathbf{x}(s)ds\right)^T(\mathbf{U}\otimes\mathbf{O})\left(\int_{t-\tau_c}^{t}\mathbf{x}(s)ds\right).
\end{aligned} \tag{21}
$$

According to Lemma 4, 21 can be written as the following:

$$
\begin{aligned}
\dot{V}_5(t) &= \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big[(\mathbf{x}_i(t)-\mathbf{x}_j(t))^T(\tau\mathbf{Z}+\tau_c\mathbf{O})(\mathbf{x}_i(t)-\mathbf{x}_j(t)) \\
&\quad +\left(\int_{t-\tau}^{t}\mathbf{x}_i(s)ds-\int_{t-\tau}^{t}\mathbf{x}_j(s)ds\right)^T\left(\frac{1}{\tau}\mathbf{Z}\right)\left(\int_{t-\tau}^{t}\mathbf{x}_i(s)ds-\int_{t-\tau}^{t}\mathbf{x}_j(s)ds\right) \\
&\quad +\left(\int_{t-\tau_c}^{t}\mathbf{x}_i(s)ds-\int_{t-\tau_c}^{t}\mathbf{x}_j(s)ds\right)^T\left(\frac{1}{\tau_c}\mathbf{O}\right)\left(\int_{t-\tau_c}^{t}\mathbf{x}_i(s)ds-\int_{t-\tau_c}^{t}\mathbf{x}_j(s)ds\right)\Big].
\end{aligned} \tag{22}
$$

Considering 8–13 and 18–22, it is straightforward to show that

$$
\dot{V}(t) \le \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\xi_{ij}^T(t)\mathbf{\Psi}_{ij}\xi_{ij}(t)\right] \tag{23}
$$

where $\mathbf{\Psi}_{ij}$ is defined in 15 and

$$
\begin{aligned}
\xi_{ij}(t) = \Big[&(\mathbf{x}_i(t)-\mathbf{x}_j(t))^T \quad (\mathbf{x}_i(t-\tau)-\mathbf{x}_j(t-\tau))^T \quad (\mathbf{x}_i(t-\tau_c)-\mathbf{x}_j(t-\tau_c))^T \quad (\mathbf{f}(\mathbf{M}\mathbf{x}_i(t))-\mathbf{f}(\mathbf{M}\mathbf{x}_j(t)))^T \cdots \\
&\cdots (\mathbf{f}(\mathbf{M}\mathbf{x}_i(t-\tau))-\mathbf{f}(\mathbf{M}\mathbf{x}_j(t-\tau)))^T \quad (\mathbf{f}(\mathbf{D}\mathbf{x}_i(t))-\mathbf{f}(\mathbf{D}\mathbf{x}_j(t)))^T \quad (\mathbf{f}(\mathbf{D}\mathbf{x}_i(t-\tau))-\mathbf{f}(\mathbf{D}\mathbf{x}_j(t-\tau)))^T \cdots \\
&\cdots \int_{t-\tau}^{t}(\mathbf{x}_i(s)-\mathbf{x}_j(s))^T ds \quad \int_{t-\tau_c}^{t}(\mathbf{x}_i(s)-\mathbf{x}_j(s))^T ds \Big]^T.
\end{aligned}
$$

If $\mathbf{\Psi}_{ij}<0$ for $\forall\, 1\le i<j\le N$, then $\dot{V}(t)<0$. From Definition 1, this implies that the system 2 has a global synchronization. This completes the proof. □

IV. ILLUSTRATIVE EXAMPLES

Example 1. Consider the system 2 with the following parameters [31]:

$$
\begin{aligned}
&\mathbf{A}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \mathbf{A}_d=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=, \mathbf{B}=\mathbf{C}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \\
&\mathbf{M}=\begin{bmatrix}3.8 & 2\\ 0.1 & 1.8\end{bmatrix}, \mathbf{D}=\begin{bmatrix}-3.5 & 1\\ 0.1 & -1.5\end{bmatrix}, \mathbf{\Gamma}_1=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}, \\
&\mathbf{\Gamma}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \mathbf{\Delta}_1=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \mathbf{\Delta}_2=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},
\end{aligned}
$$

and $\mathbf{G}^{(1)}=\mathbf{G}^{(2)}=\mathbf{e}\mathbf{e}^T-6\mathbf{I}_6$ where $\mathbf{e}=[1,1,1,1,1,1]^T$. By applying Theorem 1 into this example and solving the LMIs 4 and 5, a feasible solution is as follows:

$$
\begin{aligned}
&\mathbf{P}=\begin{bmatrix}0.9523 & 0.1125\\ 0.1125 & 0.3831\end{bmatrix}, \mathbf{Q}_{11}=\begin{bmatrix}5.8919 & 0.4678\\ 0.4678 & 3.6381\end{bmatrix}, \\
&\mathbf{Q}_{12}=\begin{bmatrix}-0.4329 & -0.0138\\ -0.1662 & -0.3086\end{bmatrix}, \mathbf{Q}_{13}=\begin{bmatrix}-0.0251 & -0.0598\\ -0.1676 & 0.1461\end{bmatrix}, \\
&\mathbf{Q}_{22}=\begin{bmatrix}0.1281 & -0.0254\\ -0.0254 & 0.2060\end{bmatrix}, \mathbf{Q}_{23}=\begin{bmatrix}0.0753 & 0.0148\\ -0.0127 & 0.0219\end{bmatrix}, \\
&\mathbf{Q}_{33}=\begin{bmatrix}0.2510 & 0.0185\\ 0.0185 & 0.2845\end{bmatrix}, \mathbf{R}=\begin{bmatrix}5.4450 & 0.4907\\ 0.4907 & 3.0325\end{bmatrix}, \\
&\mathbf{L}_1=\mathrm{diag}\{0.1326, 0.1326\}, \mathbf{L}_2=\mathrm{diag}\{0.2017, 0.2017\}, \\
&\mathbf{J}_1=\mathrm{diag}\{0.3911, 0.3911\}, \mathbf{J}_2=\mathrm{diag}\{0.4887, 0.4887\}.
\end{aligned}
$$

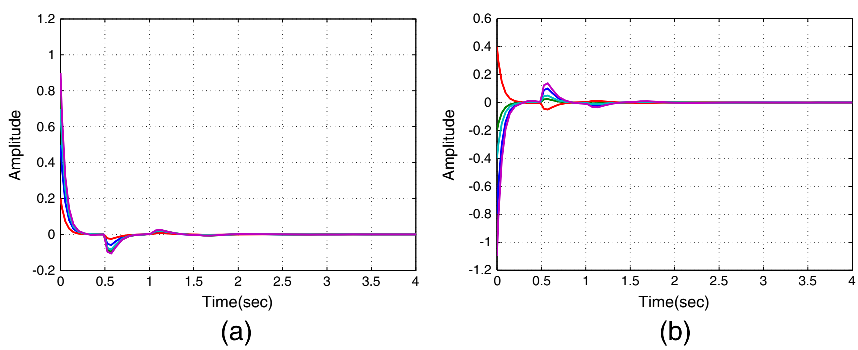

Fig. 1. Synchronization errors for networks: $[e_j(t)]$ (a) $j=1$, (b) $j=2$.

It is shown that this system can achieve global synchronization with any admissible time delay. For $f(x)=0.25(|x+1|-|x-1|)$, $\tau=0.3$, and $\tau_c=0.5$, the synchronization errors are shown in Fig. 1, where $\mathbf{e}_j(t)=(\mathbf{x}_{ij}(t)-\mathbf{x}_{1j}(t))$, $i=2,\ldots,6; j=1,2$.

Example 2. Consider the system 2 where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{D}$, $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$, $\boldsymbol{\Delta}_1$, $\mathbf{G}^{(1)}$, and $\mathbf{G}^{(2)}$ are introduced in Example 1, and

$$\mathbf{A}_d=\begin{bmatrix}-1.6 & 0.8\\ -1.2 & -2\end{bmatrix}, \boldsymbol{\Delta}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

By applying Theorem 2 into this example and solving the LMIs 15 and 16, a feasible solution is as follows:

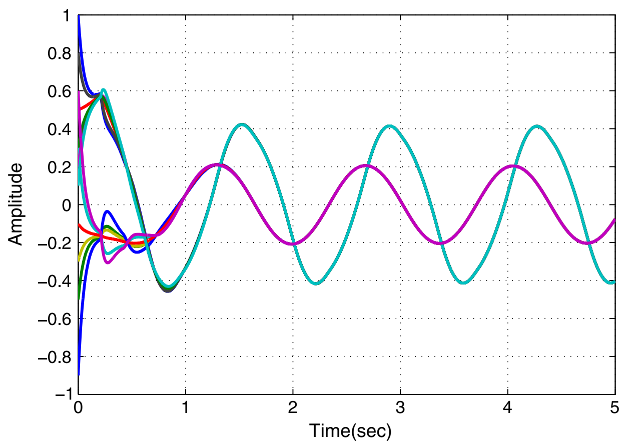

Fig. 2. State trajectories: $\mathbf{x}_i(t)$; $[i=1,2\ldots,6]$

$$\mathbf{P}=\begin{bmatrix}0.0706 & -0.1022\\ -0.1022 & 0.3904\end{bmatrix}, \mathbf{Q}_{11}=\begin{bmatrix}6.9030 & -1.1660\\ -1.1660 & 7.1221\end{bmatrix},$$
$$\mathbf{Q}_{12}=\begin{bmatrix}-0.1994 & 0.0792\\ -0.0655 & -0.5470\end{bmatrix}, \mathbf{Q}_{13}=\begin{bmatrix}0.2112 & -0.0308\\ 0.0687 & 0.0385\end{bmatrix},$$
$$\mathbf{Q}_{22}=\begin{bmatrix}0.0275 & -0.0277\\ -0.0277 & 0.1590\end{bmatrix}, \mathbf{Q}_{23}=\begin{bmatrix}0.0041 & 0.0030\\ -0.0149 & 0.0119\end{bmatrix},$$
$$\mathbf{Q}_{33}=\begin{bmatrix}0.1548 & 0.0014\\ 0.0014 & 0.1627\end{bmatrix}, \mathbf{W}_{11}=\begin{bmatrix}1.1802 & 0.0736\\ 0.0736 & 0.4458\end{bmatrix},$$
$$\mathbf{W}_{12}=\begin{bmatrix}-0.9760 & -0.2792\\ -0.2139 & -0.1595\end{bmatrix}, \mathbf{W}_{13}=\begin{bmatrix}-1.2323 & -0.3990\\ -0.1055 & -0.1642\end{bmatrix},$$
$$\mathbf{W}_{22}=\begin{bmatrix}3.0274 & 0.0608\\ 0.0608 & 2.0825\end{bmatrix}, \mathbf{W}_{23}=\begin{bmatrix}-0.9359 & 0.0562\\ 0.3442 & -0.8670\end{bmatrix},$$
$$\mathbf{W}_{33}=\begin{bmatrix}3.2938 & 0.5580\\ 0.5580 & 2.1613\end{bmatrix}, \mathbf{R}=\begin{bmatrix}5.5789 & -0.6518\\ -0.6518 & 5.7221\end{bmatrix},$$
$$\mathbf{Z}=\begin{bmatrix}7.0642 & 1.5795\\ 1.5795 & 2.5261\end{bmatrix}, \mathbf{O}=\begin{bmatrix}8.5843 & 2.3035\\ 2.3035 & 2.8893\end{bmatrix},$$
$$\mathbf{L}_1=\mathrm{diag}\{0.0257,0.0257\}, \mathbf{L}_2=\mathrm{diag}\{0.1092,0.1092\},$$
$$\mathbf{J}_1=\mathrm{diag}\{0.2927,0.2927\}, \mathbf{J}_2=\mathrm{diag}\{0.2425,0.2425\}.$$

It is shown that this system can achieve global synchronization with $\tau=\tau_c=0.21$. For $f(x)=0.5(|x+1|-|x-1|)$ and $\tau=\tau_c=0.21$ the state's trajectories and the synchronization errors are shown in Figs. 2 and 3, respectively.

V. CONCLUSION

This paper considered the problem of synchronization of complex dynamical networks with state and coupling time-delays. Based on the Lyapunov-Krasovskii functional some delay-independent and delay-dependent criteria were obtained and formulated in the form of LMIs to ascertain the synchronization between each node of the CDN. The effectiveness of the proposed method was illustrated using some numerical simulations.

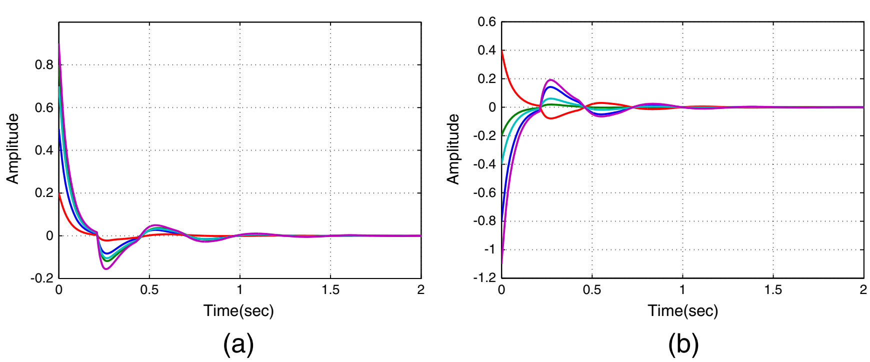

Fig. 3. Synchronization errors for networks: $[\boldsymbol{e}_j(t)]$ (a) $j=1$, (b) $j=2$.

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