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ORIGINAL ARTICLE

Global synchronization of neural networks with hybrid coupling: a delay interval segmentation approach

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Abstract The global synchronization of coupled neural networks with hybrid coupling has been studied in this paper. The hybrid coupling is formed from constant coupling, discrete-delay coupling, and distributed-delay coupling. In this regard, a larger class and more complicated coupled neural networks lead in the synchronization problem procedure. According to the new augmented Lyapunov–Krasovskii functional and the idea of *M*-segmentation of delay length, a less conservative delay-dependent criterion is obtained and expressed in the form of linear matrix inequalities. In many cases, due to the increasing segmentation number, the delay length *M*-segmentation method could give an opportunity to the user to find a bigger upper bound of the maximum allowable time delay. The effectiveness of suggested method above is proved by simulating a numerical example on a typical chaotic cellular neural network. The results show that the above-mentioned method is less conservative than the other methods reviewed in this article.

Keywords Synchronization · Neural network · Complex network · Lyapunov–Krasovskii · Time delay

1 Introduction

Since dynamical recurrent neural networks contain various applications such as pattern recognition and classification, associative memories, and dynamical systems modeling, they are likely to have inspired considerable researchers to study their behaviors [1–5]. Many phenomena in these areas such as bifurcation, chaotic behavior, and many other are studied extensively in these networks [6, 7]. On the one hand, these neural networks have the potential to be coupled together and as a result create a complex network [8]. On the other hand, the neural networks should cooperate and communicate with each other so that to create a complex network of networks [9].

In complex network science, synchronization behavior is probably one of the most noticeable phenomena among other network's nodes [10–17]. The synchronization problem between two chaotic systems was first introduced by Pecora and Carroll [18], and then, enormous attentions were guided to apply this problem in many complex networks, due to its potential applications in real-world practice such as secure communication, harmonic oscillation generation in human heartbeat regulation, and agent's synchronization in association management used to improve their work efficiency [19–24]. For instance, an architecture to store and retrieve complex oscillatory patterns in the synchronized states of a coupled neural networks is presented in [1].

Furthermore, time delay in practice for the coupling connections in the neural networks is inevitable because of the finiteness of signal transmission speed over the links. This issue causes in many difficulties such as poor performance, stability margin reduction, and increasing complexity [25–27]. Thus, synchronization of coupled

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neural networks with time delays in coupling connections seems considerably significant [21, 23, 28–30]. Moreover, the problem of networks based synchronization of delayed neural networks is discussed in [31] and references therein. In general, the time delay can be classified as discrete (lumped) time delay and distributed time delay. Most papers appear to consider discrete time delays in their coupling connections between the neural networks [23, 32–34]. In [35], it shows that introducing distributed delays can solve more problems in general pattern recognition. Therefore, both discrete and distributed time delays have been studied for the synchronization problem in coupled neural networks in [21, 36, 37]. In [21], less conservative criteria are given in comparison with those given in [36, 37]. In this paper, based on the new augmented Lyapunov–Krasovskii functional, a less conservative delay-dependent criterion is obtained and expressed in the form of linear matrix inequalities. In this method, the delay interval is discretized into M-equal segments and for each segment, and a different constant weighting matrix is considered for LKF. The main advantage of this method is better approximation of time-varying weighting matrices in LKF and giving more weighting matrices to the Lyapunov functional [38, 39].

Generally, for this problem, some criteria are given to the user as LMIs and he/she can check their feasibility for his/her own case. If these LMIs are feasible, the synchronization is guaranteed for a known maximum time delay. However, the problem arises when the user needs larger time delay due to his/her application. Usually, these criteria do not have any mechanism giving the user any choice to expand the maximum allowable time delay. The introduced criterion in this paper, in comparison with the above-mentioned methods, gave an opportunity to the user to increase the maximum allowable time delay by increasing the segmentation number of the time delay interval. In proof, an illustrative example, including a typical chaotic cellular neural network, is simulated in order to demonstrate the efficiency of the results in comparison with the other methods.

This paper is organized as follows. In Sect. 2, the problem formulations for the coupled neural networks structure along with a number of lemmas are presented. In Sect. 3, based on a new augmented LKF, a criterion is given to ascertain the synchronization between the nodes of the coupled networks. Section 4 provides the simulation results. Lastly, Sect. 5 concludes the paper.

Notations Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. $\mathbf{P} > 0$ means that $\mathbf{P}$ is a real positive definite and symmetric matrix. $\mathbf{I}$ is the identity matrix with appropriate dimensions and $\mathrm{diag}\{\mathbf{W}_1, \ldots, \mathbf{W}_m\}$ refers to a real matrix with diagonal elements $\mathbf{W}_1, \ldots, \mathbf{W}_m$. $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of the real matrix $\mathbf{A}$. Symmetric terms in a symmetric matrix are denoted by $*$, and the sign $\otimes$ is stand for the Kronecker product.

2 Problem statement and preliminaries

Consider coupled neural networks as follows:

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= -\mathbf{C}\mathbf{x}_i(t) + \mathbf{A}\mathbf{f}(\mathbf{x}_i(t)) + \mathbf{B}\mathbf{f}(\mathbf{x}_i(t-\tau)) + \sum_{j=1}^{N} G_{ij}^{(1)}\boldsymbol{\Gamma}_1\mathbf{x}_j(t) \\ &\quad + \sum_{j=1}^{N} G_{ij}^{(2)}\boldsymbol{\Gamma}_2\mathbf{x}_j(t-\tau) + \sum_{j=1}^{N} G_{ij}^{(3)}\boldsymbol{\Gamma}_3\int_{t-\tau}^{t}\mathbf{x}_j(s)\mathrm{d}s, \\ &\quad i = 1,2,\ldots,N \end{aligned} \tag{1}$$

where $\mathbf{x}_i(t) = [x_{i1}(t) \quad x_{i2}(t) \quad \cdots \quad x_{in}(t)]^{\mathrm{T}} \in \mathbb{R}^n$ denote the state vector of the ith neural network, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is the neuron activation function, $\mathbf{C}, \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are constant matrices, and $\tau > 0$ denotes the state delay. $\mathbf{G}^{(q)} = (G_{ij}^{(q)})_{N\times N}$, $(q = 1,2,3)$ denotes the coupling connections and $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3 \in \mathbb{R}^{n\times n}$ represent the inner coupling matrices. It is assumed that the discrete-delay and the distributed-delay are identical [36, 37].

Assumption 1 The coupling connection matrices should satisfy

$$\begin{cases} G_{ij}^{(q)} = G_{ji}^{(q)} \geq 0, & i \neq j,\ q = 1,2,3, \\ G_{ii}^{(q)} = -\sum\limits_{j=1, j\neq i}^{N} G_{ij}^{(q)} \geq 0, & i,j = 1,\ldots,N,\ q = 1,2,3. \end{cases}$$

Throughout this paper, the following assumption on $\mathbf{f}(.)$ is made.

Assumption 2 For any $x_1, x_2 \in \mathbb{R}$, there are some constants, σ_r^-, σ_r^+, which the nonlinear function satisfies.

$$\sigma_r^- \leq \frac{f_r(x_1) - f_r(x_2)}{x_1 - x_2} \leq \sigma_r^+, \quad r = 1,2,\ldots,n.$$

Definition 1 The coupled neural networks (1) is said to be globally synchronized for any initial conditions $\Pi_{i0}(s)$, $(i = 1, 2, \ldots, N)$, such that $\lim_{t\to\infty}\|\mathbf{e}_i(t)\| = \lim_{t\to\infty} \|\mathbf{x}_i(t) - \mathbf{s}(t)\| = 0$, $(i = 1, 2, \ldots, N)$, where $\|.\|$ stands for the Euclidean vector norm and $\mathbf{s}(t) \in \mathbb{R}^n$ is a synchronization manifold, which can be either an equilibrium point, a periodic orbit, or an orbit of a chaotic attractor and satisfies

$$\dot{\mathbf{s}}(t) = -\mathbf{C}\mathbf{s}(t) + \mathbf{A}\mathbf{f}(\mathbf{s}(t)) + \mathbf{B}\mathbf{f}(\mathbf{s}(t-\tau)). \tag{2}$$

With defining error vectors as follows:

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \quad i = 1, 2, \ldots, N, \tag{3}$$

and substituting in (1), the error dynamics would be

$$\begin{aligned}\dot{\mathbf{e}}_i(t) = & -\mathbf{C}\mathbf{e}_i(t) + \mathbf{A}\mathbf{f}(\mathbf{e}_i(t)) + \mathbf{B}\mathbf{f}(\mathbf{e}_i(t-\tau)) \\ & + \sum_{j=1}^{N} G_{ij}^{(1)}\boldsymbol{\Gamma}_1\mathbf{e}_j(t) + \sum_{j=1}^{N} G_{ij}^{(2)}\boldsymbol{\Gamma}_2\mathbf{e}_j(t-\tau) \\ & + \sum_{j=1}^{N} G_{ij}^{(3)}\boldsymbol{\Gamma}_3\int_{t-\tau}^{t}\mathbf{e}_j(s)\mathrm{d}s, \quad i = 1, 2, \ldots, N,\end{aligned} \tag{4}$$

where $\mathbf{f}(\mathbf{e}_i(t)) = \mathbf{f}(\mathbf{x}_i(t)) - \mathbf{f}(\mathbf{s}(t))$ and $\mathbf{f}(\mathbf{e}_i(t-\tau)) = \mathbf{f}(\mathbf{x}_i(t-\tau)) - \mathbf{f}(\mathbf{s}(t-\tau))$.

For notation simplicity, let

$$\begin{aligned}\mathbf{e}(t) &= \begin{bmatrix}\mathbf{e}_1^{\mathrm{T}}(t) & \mathbf{e}_2^{\mathrm{T}}(t) & \cdots & \mathbf{e}_N^{T}(t)\end{bmatrix}^{\mathrm{T}}, \ \mathbf{F}(\mathbf{e}(t)) \\ &= \begin{bmatrix}\mathbf{f}^{\mathrm{T}}(\mathbf{e}_1(t)) & \mathbf{f}^{\mathrm{T}}(\mathbf{e}_2(t)) & \cdots & \mathbf{f}^{\mathrm{T}}(\mathbf{e}_N(t))\end{bmatrix}^{\mathrm{T}}.\end{aligned}$$

With the help of the matrix Kronecker product, the coupled neural network (4) can be written as the following form:

$$\begin{aligned}\dot{\mathbf{e}}(t) = & -(\mathbf{I}_N\otimes\mathbf{C})\mathbf{e}(t) + (\mathbf{I}_N\otimes\mathbf{A})\mathbf{F}(\mathbf{e}(t)) + (\mathbf{I}_N\otimes\mathbf{B})\mathbf{F}(\mathbf{e}(t-\tau)) \\ & + \left(\mathbf{G}^{(1)}\otimes\boldsymbol{\Gamma}_1\right)\mathbf{e}(t) + \left(\mathbf{G}^{(2)}\otimes\boldsymbol{\Gamma}_2\right)\mathbf{e}(t-\tau) \\ & + \left(\mathbf{G}^{(3)}\otimes\boldsymbol{\Gamma}_3\right)\int_{t-\tau}^{t}\mathbf{e}(s)\mathrm{d}s.\end{aligned} \tag{5}$$

The following lemmas are needed in the derivations of the main results.

Lemma 1 ((Jensen Inequality), [14]) *Assume that the vector function* $\boldsymbol{\omega} : [0, r] \to \mathbb{R}^n$ *is well defined for the following integrations. For any symmetric matrix* $\mathbf{R} \in \mathbb{R}^{n\times n}$ *and scalar* $r > 0$, *one has*

$$r\int_0^r \boldsymbol{\omega}^{\mathrm{T}}(s)\mathbf{R}\omega(s)\mathrm{d}s \geq \left(\int_0^r \boldsymbol{\omega}(s)\mathrm{d}s\right)^{\mathrm{T}}\mathbf{R}\left(\int_0^r \boldsymbol{\omega}(s)\mathrm{d}s\right).$$

Lemma 2 *According to* [40] *and Assumption 2, for any diagonal matrices* $\mathbf{J} > 0, \mathbf{L} > 0$, *it follows that:*

$$\begin{aligned}&\boldsymbol{\theta}^{\mathrm{T}}(t)\begin{bmatrix}-\mathbf{J}\Delta_1 & \mathbf{J}\Delta_2 \\ * & -\mathbf{J}\end{bmatrix}\boldsymbol{\theta}(t) \\ &+ \boldsymbol{\theta}^{\mathrm{T}}(t-\tau)\begin{bmatrix}-\mathbf{L}\Delta_1 & \mathbf{L}\Delta_2 \\ * & -\mathbf{L}\end{bmatrix}\boldsymbol{\theta}(t-\tau) \geq 0,\end{aligned} \tag{6}$$

where,

$$\boldsymbol{\theta}(t) = \begin{bmatrix}\mathbf{e}_i(t) - \mathbf{e}_j(t) \\ \mathbf{f}(\mathbf{e}_i(t)) - \mathbf{f}(\mathbf{e}_j(t))\end{bmatrix},$$
$$\boldsymbol{\Delta}_1 = \mathrm{diag}\left[\sigma_1^+\sigma_1^-, \ldots, \sigma_n^+\sigma_n^-\right],$$
$$\boldsymbol{\Delta}_2 = \mathrm{diag}\left[\frac{\sigma_1^+ + \sigma_1^-}{2}, \ldots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right].$$

Lemma 3 ([23]) *Let* $\mathbf{1} = [1, 1, \ldots, 1]^{\mathrm{T}}$, $\mathbf{E}_N = \mathbf{1}\mathbf{1}^{\mathrm{T}}$, *and* $\mathbf{U} = N\mathbf{I}_N - \mathbf{E}_N$, $\mathbf{P} \in \mathbb{R}^{n\times n}$, $\mathbf{x} = \left[\mathbf{x}_1^{\mathrm{T}}, \ldots, \mathbf{x}_N^{\mathrm{T}}\right]^{\mathrm{T}}$, *and* $\mathbf{y} = \left[\mathbf{y}_1^{\mathrm{T}}, \ldots, \mathbf{y}_N^{\mathrm{T}}\right]^{\mathrm{T}}$ *with* $\mathbf{x}_k, \mathbf{y}_k \in \mathbb{R}^n$, $(k = 1, 2, \ldots, N)$, *then*

$$\mathbf{x}^{\mathrm{T}}(\mathbf{U}\otimes\mathbf{P})\mathbf{y} = \sum_{1\leq i<j\leq N}^{N}\left(\mathbf{x}_i - \mathbf{x}_j\right)^{\mathrm{T}}\mathbf{P}\left(\mathbf{y}_i - \mathbf{y}_j\right).$$

3 Main results

In this section, stability of the error dynamic (5) which guarantees the synchronization between the neural networks in (1) is discussed.

Theorem 1 *For given* τ *and* m, *the coupled neural networks* (1) *is globally synchronized if there exist positive definite matrices* $\mathbf{Q}^{(k)} > 0$, $\mathbf{W} > 0$, $\mathbf{R}^{(k)} > 0$, *real matrices* $\mathbf{T}_q$, $\mathbf{P}_q$, $\mathbf{Y}_k$, $\mathbf{H}_k$, $q = 1, \ldots, 5$, $k = 1, \ldots, m$, *and positive diagonal matrices* $\mathbf{J} > 0$, $\mathbf{L} > 0$, *such that the following LMIs hold for all* $1 \leq i < j \leq N$:

$$\begin{aligned}\boldsymbol{\Psi}_{ij} &= \begin{bmatrix}\Pi^{(1,1)} & \Pi^{(1,2)} & \cdots & \Pi^{(1,10)} \\ * & \Pi^{(2,2)} & \cdots & \Pi^{(2,10)} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Pi^{(10,10)}\end{bmatrix} < 0, \\ \mathbf{W} &= \begin{bmatrix}\mathbf{Z} & \mathbf{O}^{(1)} & \mathbf{O}^{(2)} \\ * & \mathbf{W}^{(1)} & \mathbf{W}^{(2)} \\ * & * & \mathbf{W}^{(3)}\end{bmatrix} > 0, \ \mathbf{Q}^{(k)} \\ &= \begin{bmatrix}\mathbf{Q}_{11}^{(k)} & \mathbf{Q}_{12}^{(k)} & \mathbf{Q}_{13}^{(k)} \\ * & \mathbf{Q}_{22}^{(k)} & \mathbf{Q}_{23}^{(k)} \\ * & * & \mathbf{Q}_{33}^{(k)}\end{bmatrix} > 0, \\ \mathbf{R}^{(k)} &= \begin{bmatrix}\mathbf{R}_{11}^{(k)} & \mathbf{R}_{12}^{(k)} & \mathbf{R}_{13}^{(k)} \\ * & \mathbf{R}_{22}^{(k)} & \mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{R}_{33}^{(k)}\end{bmatrix} > 0, \quad k = 1, \ldots, m,\end{aligned} \tag{7}$$

where,

$$\mathbf{O}^{(1)} = \begin{bmatrix} \mathbf{O}_1^{(1)} & \mathbf{O}_2^{(1)} & \cdots & \mathbf{O}_m^{(1)} \end{bmatrix}, \mathbf{O}^{(2)} = \begin{bmatrix} \mathbf{O}_1^{(2)} & \mathbf{O}_2^{(2)} & \cdots & \mathbf{O}_m^{(2)} \end{bmatrix}, \mathbf{W}^{(q)} = \begin{bmatrix} \mathbf{W}_{11}^{(q)} & \mathbf{W}_{12}^{(q)} & \cdots & \mathbf{W}_{1m}^{(q)} \\ * & \mathbf{W}_{22}^{(q)} & \cdots & \mathbf{W}_{2m}^{(q)} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \mathbf{W}_{mm}^{(q)} \end{bmatrix}, (q = 1,3),$$

$$\mathbf{W}^{(2)} = \begin{bmatrix} \mathbf{W}_{11}^{(2)} & \mathbf{W}_{12}^{(2)} & \cdots & \mathbf{W}_{1m}^{(2)} \\ \mathbf{W}_{21}^{(2)} & \mathbf{W}_{22}^{(2)} & \cdots & \mathbf{W}_{2m}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{W}_{m1}^{(2)}* & \mathbf{W}_{m2}^{(2)} & \cdots & \mathbf{W}_{mm}^{(2)} \end{bmatrix},$$

$$\Pi^{(1,1)} = \sum_{k=1}^{m} \left(-h\mathbf{Q}_{11}^{(k)}\mathbf{C} - h\mathbf{C}^T\mathbf{Q}_{11}^{(k)} - NhG_{ij}^{(1)} \left(\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_1 + \mathbf{\Gamma}_1^T\mathbf{Q}_{11}^{(k)} \right) + h^2\mathbf{R}_{11}^{(k)} \right) + \mathbf{Q}_{12}^{(1)} + \left(\mathbf{Q}_{12}^{(1)} \right)^{\mathrm{T}} + \mathbf{Q}_{22}^{(1)}$$

$$+ \mathbf{O}_1^{(1)} + \left(\mathbf{O}_1^{(1)} \right)^{\mathrm{T}} - \mathbf{J}\Delta_1 - \mathbf{T}_2\mathbf{C} - \mathbf{C}^{\mathrm{T}}\mathbf{T}_2^{\mathrm{T}} - NG_{ij}^{(1)} \left(\mathbf{T}_2\Gamma_1 + \Gamma_1^{\mathrm{T}}\mathbf{T}_2^{\mathrm{T}} \right) + \mathbf{P}_2 + \mathbf{P}_2^{\mathrm{T}},$$

$$\Pi^{(1,2)} = \begin{bmatrix} \Pi_1^{(1,2)} & \cdots & \Pi_{m-1}^{(1,2)} \end{bmatrix}, \Pi_k^{(1,2)} = \mathbf{Q}_{12}^{(k+1)} - \mathbf{Q}_{12}^{(k)} - \mathbf{O}_k^{(1)} + \mathbf{O}_{k+1}^{(1)},$$

$$\Pi^{(1,3)} = \sum_{k=1}^{m} \left(-NhG_{ij}^{(2)}\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_2 \right) - \mathbf{Q}_{12}^{(m)} - \mathbf{O}_m^{(1)} - NG_{ij}^{(2)}\mathbf{T}_2\Gamma_2 - \mathbf{P}_2 - \mathbf{C}^{\mathrm{T}}\mathbf{T}_4^{\mathrm{T}} - NG_{ij}^{(1)}\mathbf{\Gamma}_1^{\mathrm{T}}\mathbf{T}_4^{\mathrm{T}} + \mathbf{P}_4^{\mathrm{T}},$$

$$\Pi^{(1,4)} = \sum_{k=1}^{m} \left(hQ_{11}^{(k)}A + h^2\mathbf{R}_{13}^{(k)} \right) + \mathbf{Q}_{23}^{(1)} + \mathbf{Q}_{13}^{(1)} + \mathbf{J}\Delta_2 + \mathbf{O}_1^{(2)} + \mathbf{T}_2\mathbf{A} - \mathbf{C}^{\mathrm{T}}\mathbf{T}_3^{\mathrm{T}} - NG_{ij}^{(1)}\mathbf{\Gamma}_1^{\mathrm{T}}\mathbf{T}_3^{\mathrm{T}} + \mathbf{P}_3^{\mathrm{T}},$$

$$\Pi^{(1,5)} = \begin{bmatrix} \Pi_1^{(1,5)} & \cdots & \Pi_{m-1}^{(1,5)} \end{bmatrix}, \Pi_k^{(1,5)} = \mathbf{Q}_{13}^{(k+1)} - \mathbf{Q}_{13}^{(k)} + \mathbf{O}_{k+1}^{(2)} - \mathbf{O}_k^{(2)},$$

$$\Pi^{(1,6)} = \sum_{k=1}^{m} \left(h\mathbf{Q}_{11}^{(k)}\mathbf{B} \right) - \mathbf{Q}_{13}^{(m)} - \mathbf{O}_m^{(2)} + \mathbf{T}_2\mathbf{B} - \mathbf{C}^{\mathrm{T}}\mathbf{T}_5^{\mathrm{T}} - NG_{ij}^{(1)}\mathbf{\Gamma}_1^{\mathrm{T}}\mathbf{T}_5^{\mathrm{T}} + \mathbf{P}_5^{\mathrm{T}},$$

$$\Pi^{(1,7)} = \begin{bmatrix} \Pi_1^{(1,7)} & \cdots & \Pi_m^{(1,7)} \end{bmatrix}, \Pi_k^{(1,7)} = -hNG_{ij}^{(3)}\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_3 - NG_{ij}^{(3)}\mathbf{T}_2\mathbf{\Gamma}_3 + \mathbf{W}_{1k}^{(1)},$$

$$\Pi^{(1,8)} = \begin{bmatrix} \Pi_1^{(1,8)} & \cdots & \Pi_m^{(1,8)} \end{bmatrix}, \Pi_k^{(1,8)} = -\mathbf{C}^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}} - NG_{ij}^{(1)}\mathbf{\Gamma}_1^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}},$$

$$\Pi^{(1,9)} = \sum_{k=1}^{m} \left(h^2R_{12}^{(k)} \right) - \mathbf{C}^{\mathrm{T}}\mathbf{T}_1^{\mathrm{T}} + \mathbf{Z} - NG_{ij}^{(1)}\mathbf{\Gamma}_1^{\mathrm{T}}\mathbf{T}_1^{\mathrm{T}} - \mathbf{T}_2 + \mathbf{P}_1^{\mathrm{T}},$$

$$\Pi^{(1,10)} = \begin{bmatrix} \Pi_1^{(1,10)} & \cdots & \Pi_m^{(1,10)} \end{bmatrix}, \Pi_k^{(1,8)} = -\mathbf{P}_2 + \mathbf{Y}_k^{\mathrm{T}},$$

$$\Pi^{(2,2)} = \mathrm{diag}\left\{ \Pi_1^{(2,2)}, \ldots, \Pi_{m-1}^{(2,2)} \right\}, \ \Pi_k^{(2,2)} = \mathbf{Q}_{22}^{(k+1)} - \mathbf{Q}_{22}^{(k)},$$

$$\Pi^{(2,5)} = \mathrm{diag}\left\{ \Pi_1^{(2,5)}, \ldots, \Pi_{m-1}^{(2,5)} \right\}, \ \Pi_k^{(2,5)} = \mathbf{Q}_{23}^{(k+1)} - \mathbf{Q}_{23}^{(k)},$$

$$\Pi^{(2,7)} = \begin{bmatrix} \left(\mathbf{W}_{12}^{(1)} \right)^{\mathrm{T}} - \mathbf{W}_{11}^{(1)} & \mathbf{W}_{22}^{(1)} - \mathbf{W}_{12}^{(1)} & \cdots & \mathbf{W}_{2m}^{(1)} - \mathbf{W}_{1m}^{(1)} \\ \left(\mathbf{W}_{13}^{(1)} \right)^{\mathrm{T}} - \left(\mathbf{W}_{12}^{(1)} \right)^{\mathrm{T}} & \left(\mathbf{W}_{23}^{(1)} \right)^{\mathrm{T}} - \mathbf{W}_{22}^{(1)} & \cdots & \mathbf{W}_{3m}^{(1)} - \mathbf{W}_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mathbf{W}_{1m}^{(1)} \right)^{\mathrm{T}} - \left(\mathbf{W}_{1(m-1)}^{(1)} \right)^{\mathrm{T}} & \left(\mathbf{W}_{2m}^{(1)} \right)^{\mathrm{T}} - \left(\mathbf{W}_{2(m-1)}^{(1)} \right)^{\mathrm{T}} & \cdots & \mathbf{W}_{mm}^{(1)} - \mathbf{W}_{(m-1)m}^{(1)} \end{bmatrix},$$

$$\Pi^{(3,3)} = -\mathbf{Q}_{22}^{(m)} - \mathbf{L}\Delta_1 - NG_{ij}^{(2)} \left(\mathbf{T}_4\Gamma_2 + \Gamma_2^{\mathrm{T}}\mathbf{T}_4^{\mathrm{T}} \right) - \mathbf{P}_4 - \mathbf{P}_4^{\mathrm{T}}, \ \Pi^{(3,4)} = -NG_{ij}^{(2)}\Gamma_2^{\mathrm{T}}\mathbf{T}_3^{\mathrm{T}} - \mathbf{P}_3^{\mathrm{T}} + \mathbf{T}_4\mathbf{A},$$

$$\Pi^{(3,6)} = -\mathbf{Q}_{23}^{(m)} + \mathbf{L}\Delta_2 + \mathbf{T}_4\mathbf{B} - NG_{ij}^{(2)}\Gamma_2^{\mathrm{T}}\mathbf{T}_5^{\mathrm{T}} - \mathbf{P}_5^{\mathrm{T}}, \ \Pi^{(3,7)} = \begin{bmatrix} \Pi_1^{(3,7)} & \cdots & \Pi_m^{(3,7)} \end{bmatrix}, \ \Pi_k^{(3,7)} = -NG_{ij}^{(3)}\Gamma_3^{\mathrm{T}}\mathbf{T}_4^{\mathrm{T}} - \left(W_{km}^{(1)} \right)^{\mathrm{T}},$$

$$\Pi^{(3,8)} = \begin{bmatrix} \Pi_1^{(3,8)} & \cdots & \Pi_m^{(3,8)} \end{bmatrix}, \ \Pi_k^{(3,8)} = -NG_{ij}^{(2)}\Gamma_2^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}}, \ \Pi^{(3,9)} = -NG_{ij}^{(2)}\Gamma_2^{\mathrm{T}}\mathbf{T}_1^{\mathrm{T}} - \mathbf{P}_1^{\mathrm{T}} - \mathbf{T}_4,$$

$$\Pi^{(3,10)} = \left[\Pi_1^{(3,10)} \quad \cdots \quad \Pi_m^{(3,10)} \right],\ \Pi_k^{(3,10)} = -\mathbf{Y}_k^{\mathrm{T}} - \mathbf{P}_4,\ \Pi^{(4,4)} = \sum_{k=1}^{m} \left(h^2 \mathbf{R}_{33}^{(k)} \right) + \mathbf{Q}_{33}^{(1)} - \mathbf{J} + \mathbf{T}_3\mathbf{A} + \mathbf{A}\mathbf{T}_3^{\mathrm{T}},$$

$$\Pi^{(4,6)} = \mathbf{T}_3\mathbf{B} + \mathbf{A}^{\mathrm{T}}\mathbf{T}_5^{\mathrm{T}}, \Pi^{(4,7)} = \left[\Pi_1^{(4,7)} \quad \cdots \quad \Pi_m^{(4,7)} \right],\ \Pi_k^{(4,7)} = -NG_{ij}^{(3)}\mathbf{T}_3\Gamma_3 + \left(\mathbf{W}_{k1}^{(2)} \right)^{\mathrm{T}},$$

$$\Pi^{(4,8)} = \left[\Pi_1^{(4,8)} \quad \cdots \quad \Pi_m^{(4,8)} \right],\ \Pi_k^{(4,8)} = \mathbf{A}^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}} + \mathbf{W}_{1k}^{(3)},\ \Pi^{(4,9)} = \sum_{k=1}^{m} \left(h^2 \left(\mathbf{R}_{23}^{(k)} \right)^{\mathrm{T}} \right) + \mathbf{A}^{\mathrm{T}}\mathbf{T}_1^{T} - \mathbf{T}_3,$$

$$\Pi^{(4,10)} = -[\mathbf{P}_3 \quad \cdots \quad \mathbf{P}_3],\ \Pi^{(5,5)} = \mathrm{diag}\left\{ \Pi_1^{(5,5)}, \ldots, \Pi_{m-1}^{(5,5)} \right\},\ \Pi_k^{(5,5)} = \mathbf{Q}_{33}^{(k+1)} - \mathbf{Q}_{33}^{(k)},$$

$$\Pi^{(5,7)} = \begin{bmatrix} \left(\mathbf{W}_{12}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{11}^{(2)}\right)^{\mathrm{T}} & \left(\mathbf{W}_{22}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{21}^{(2)}\right)^{\mathrm{T}} & \cdots & \left(\mathbf{W}_{m2}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{m1}^{(2)}\right)^{\mathrm{T}} \\ \left(\mathbf{W}_{13}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{12}^{(2)}\right)^{\mathrm{T}} & \left(\mathbf{W}_{23}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{22}^{(2)}\right)^{\mathrm{T}} & \cdots & \left(\mathbf{W}_{m3}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{m2}^{(2)}\right)^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mathbf{W}_{1m}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{1(m-1)}^{(2)}\right)^{\mathrm{T}} & \left(\mathbf{W}_{2m}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{2(m-1)}^{(2)}\right)^{\mathrm{T}} & \cdots & \left(\mathbf{W}_{mm}^{(2)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{m(m-1)}^{(2)}\right)^{\mathrm{T}} \end{bmatrix},$$

$$\Pi^{(5,8)} = \begin{bmatrix} \left(\mathbf{W}_{12}^{(3)}\right)^{\mathrm{T}} - \mathbf{W}_{11}^{(3)} & \mathbf{W}_{22}^{(3)} - \mathbf{W}_{12}^{(3)} & \cdots & \mathbf{W}_{2m}^{(3)} - \mathbf{W}_{1m}^{(3)} \\ \left(\mathbf{W}_{13}^{(3)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{12}^{(3)}\right)^{\mathrm{T}} & \left(\mathbf{W}_{23}^{(3)}\right)^{\mathrm{T}} - \mathbf{W}_{22}^{(3)} & \cdots & \mathbf{W}_{3m}^{(3)} - \mathbf{W}_{2m}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mathbf{W}_{1m}^{(3)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{1(m-1)}^{(3)}\right)^{\mathrm{T}} & \left(\mathbf{W}_{2m}^{(3)}\right)^{\mathrm{T}} - \left(\mathbf{W}_{2(m-1)}^{(3)}\right)^{\mathrm{T}} & \cdots & \mathbf{W}_{mm}^{(3)} - \mathbf{W}_{(m-1)m}^{(3)} \end{bmatrix},$$

$$\Pi^{(6,6)} = -\mathbf{Q}_{33}^{(m)} - \mathbf{L} + \mathbf{T}_5\mathbf{B} + \mathbf{B}^{\mathrm{T}}\mathbf{T}_5^{\mathrm{T}},\ \Pi^{(6,7)} = \left[\Pi_1^{(6,7)} \quad \cdots \quad \Pi_m^{(6,7)} \right],\ \Pi_k^{(6,7)} = -NG_{ij}^{(3)}\mathbf{T}_5\Gamma_3 - \left(\mathbf{W}_{km}^{(2)} \right)^{\mathrm{T}}$$

$$\Pi^{(6,8)} = \left[\Pi_1^{(6,8)} \quad \cdots \quad \Pi_m^{(6,8)} \right],\ \Pi_k^{(6,8)} = \mathbf{B}^{\mathrm{T}}\mathbf{H}_k^{\mathrm{T}} - \left(\mathbf{W}_{km}^{(3)} \right)^{\mathrm{T}},\ \Pi^{(6,9)} = \mathbf{B}^{\mathrm{T}}\mathbf{T}_1^{\mathrm{T}} - \mathbf{T}_5,$$

$$\Pi^{(6,10)} = -[\mathbf{P}_5 \quad \cdots \quad \mathbf{P}_5],\ \Pi^{(7,7)} = -\mathrm{diag}\left\{ \mathbf{R}_{11}^{(1)}, \ldots, \mathbf{R}_{11}^{(m)} \right\},$$

$$\Pi^{(7,8)} = -\mathrm{diag}\left\{ \mathbf{R}_{13}^{(1)}, \ldots, \mathbf{R}_{13}^{(m)} \right\} - [\underbrace{1 \quad \cdots \quad 1}_{m}]^{\mathrm{T}} \left[NG_{ij}^{(3)}\boldsymbol{\Gamma}_3^{\mathrm{T}}\mathbf{H}_1^{\mathrm{T}} \quad \cdots \quad NG_{ij}^{(3)}\boldsymbol{\Gamma}_3^{\mathrm{T}}\mathbf{H}_m^{\mathrm{T}} \right],$$

$$\Pi^{(7,9)} = \left[\Pi_1^{(7,9)} \quad \cdots \quad \Pi_m^{(7,9)} \right]^{\mathrm{T}},\ \Pi_k^{(7,9)} = \mathbf{Q}_{12}^{(k)} - NG_{ij}^{(3)}\mathbf{T}_1\boldsymbol{\Gamma}_3,\ \Pi^{(7,10)} = -\mathrm{diag}\left\{ \mathbf{R}_{12}^{(1)}, \ldots, \mathbf{R}_{12}^{(m)} \right\},$$

$$\Pi^{(8,8)} = -\mathrm{diag}\left\{ \mathbf{R}_{33}^{(1)}, \ldots, \mathbf{R}_{33}^{(m)} \right\},\ \Pi^{(8,9)} = \left[\Pi_1^{(8,9)} \quad \cdots \quad \Pi_m^{(8,9)} \right]^{\mathrm{T}},\ \Pi_k^{(8,9)} = \mathbf{Q}_{13}^{(k)} - \mathbf{H}_k^{\mathrm{T}},$$

$$\Pi^{(8,10)} = -\mathrm{diag}\left\{ \mathbf{R}_{23}^{(1)}, \ldots, \mathbf{R}_{23}^{(m)} \right\},\ \Pi^{(9,9)} = \sum_{k=1}^{m} \left(h^2 \mathbf{R}_{22}^{(k)} \right) - \mathbf{T}_1 - \mathbf{T}_1^{\mathrm{T}},\ \Pi^{(9,10)} = -[\mathbf{P}_1 \quad \cdots \quad \mathbf{P}_1],$$

$$\Pi^{(10,10)} = -\mathrm{diag}\left\{ \mathbf{R}_{22}^{(1)}, \ldots, \mathbf{R}_{22}^{(m)} \right\} - \left[\mathbf{Y}_1^{\mathrm{T}} \quad \cdots \quad \mathbf{Y}_m^{\mathrm{T}} \right]^{\mathrm{T}} [\underbrace{1 \quad \cdots \quad 1}_{m}] - [\underbrace{1 \quad \cdots \quad 1}_{m}]^{\mathrm{T}} \left[\mathbf{Y}_1^{\mathrm{T}} \quad \cdots \quad \mathbf{Y}_m^{\mathrm{T}} \right],$$

and the other elements in the matrix $\boldsymbol{\Psi}_{ij}$ *in* (7) *which is not defined above are zero matrices with appropriate dimension.*

Proof Consider the following Lyapunov–Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{8}$$

where,

$$V_1(t) = \sum_{k=1}^{m} \int_{t-kh}^{t-(k-1)h} \begin{bmatrix} \mathbf{e}(t) \\ \mathbf{e}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{Q}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{Q}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{Q}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{Q}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{Q}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{Q}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \mathbf{e}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix} \mathrm{d}s,$$

$$V_2(t) = \begin{bmatrix} \mathbf{e}(t) \\ \Upsilon \\ \Theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{Z} & \mathbf{U}\otimes\mathbf{O}^{(1)} & \mathbf{U}\otimes\mathbf{O}^{(2)} \\ * & \mathbf{U}\otimes\mathbf{W}^{(1)} & \mathbf{U}\otimes\mathbf{W}^{(2)} \\ * & * & \mathbf{U}\otimes\mathbf{W}^{(3)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \Upsilon \\ \Theta \end{bmatrix},$$

$$V_3(t) = \sum_{k=1}^{m} \int_{-kh}^{-(k-1)h} \int_{t+\theta}^{t} h \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{R}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{R}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{R}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix} \mathrm{d}s\mathrm{d}\theta.$$

where $\mathbf{U}$ is defined in Lemma 3, $h = \tau/m$, and

$$\Upsilon = \left[\int_{t-h}^{t} \mathbf{e}^{\mathrm{T}}(s)\mathrm{d}s \quad \int_{t-2h}^{t-h} \mathbf{e}^{\mathrm{T}}(s)\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h} \mathbf{e}^{\mathrm{T}}(s)\mathrm{d}s \right]^{\mathrm{T}},$$
$$\Theta = \left[\int_{t-h}^{t} \mathbf{F}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \int_{t-2h}^{t-h} \mathbf{F}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h} \mathbf{F}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \right]^{\mathrm{T}}.$$

Remark 1 The segmentation number m should be selected incrementally. Start choosing from $m = 1$ and then keep increasing it until the maximum allowable time delay (τ) does not improve significantly. Note that increasing m causes computational burden.

For the rest of the proof, let:

$$\mathbf{e}_{ij}(t) = \mathbf{e}_i(t) - \mathbf{e}_j(t),\ \dot{\mathbf{e}}_{ij}(t) = \dot{\mathbf{e}}_i(t) - \dot{\mathbf{e}}_j(t),$$
$$\mathbf{f}_{ij}(\mathbf{e}(t)) = \mathbf{f}(\mathbf{e}_i(t)) - \mathbf{f}(\mathbf{e}_j(t)),$$
$$\int_{t-\theta_1}^{t-\theta_2} \mathbf{e}_{ij}(s)\mathrm{d}s = \int_{t-\theta_1}^{t-\theta_2} \mathbf{e}_i(s)\mathrm{d}s - \int_{t-\theta_1}^{t-\theta_2} \mathbf{e}_j(s)\mathrm{d}s,$$
$$\int_{t-\theta_1}^{t-\theta_2} \dot{\mathbf{e}}_{ij}(s)\mathrm{d}s = \int_{t-\theta_1}^{t-\theta_2} \dot{\mathbf{e}}_i(s)\mathrm{d}s - \int_{t-\theta_1}^{t-\theta_2} \dot{\mathbf{e}}_j(s)\mathrm{d}s,$$
$$\int_{t-\theta_1}^{t-\theta_2} \mathbf{f}_{ij}(\mathbf{e}(s))\mathrm{d}s = \int_{t-\theta_1}^{t-\theta_2} \mathbf{f}(\mathbf{e}_i(s))\mathrm{d}s - \int_{t-\theta_1}^{t-\theta_2} \mathbf{f}(\mathbf{e}_j(s))\mathrm{d}s.$$

Taking the derivative of $V_1(t)$ in (8) with respect to t yields:

$$\begin{aligned}\dot{V}_1(t) = \sum_{k=1}^{m} \Big\{ & 2h\mathbf{e}^{\mathrm{T}}(t)\left(\mathbf{U}\otimes\mathbf{Q}_{11}^{(k)}\right)\dot{\mathbf{e}}(t) + 2\dot{\mathbf{e}}^{\mathrm{T}}(t)\left(\mathbf{U}\otimes\mathbf{Q}_{12}^{(k)}\right) \\ & \int_{t-kh}^{t-(k-1)h} \mathbf{e}(s)\mathrm{d}s + 2\dot{\mathbf{e}}^{\mathrm{T}}(t)\left(\mathbf{U}\otimes\mathbf{Q}_{13}^{(k)}\right)\int_{t-kh}^{t-(k-1)h} \mathbf{F}(\mathbf{e}(s))\mathrm{d}s \\ & 2\mathbf{e}^{\mathrm{T}}(t)\left(\mathbf{U}\otimes\mathbf{Q}_{12}^{(k)}\right)(\mathbf{e}(t-(k-1)h) - \mathbf{e}(t-kh)) \\ & + 2\mathbf{e}^{\mathrm{T}}(t)\left(\mathbf{U}\otimes\mathbf{Q}_{13}^{(k)}\right)(\mathbf{F}(\mathbf{e}(t-(k-1)h)) - \mathbf{F}(\mathbf{e}(t-kh))) \\ & + \mathbf{e}^{\mathrm{T}}(t-(k-1)h)\left(\mathbf{U}\otimes\mathbf{Q}_{22}^{(k)}\right)\mathbf{e}(t-(k-1)h) \\ & - \mathbf{e}^{\mathrm{T}}(t-kh)\left(\mathbf{U}\otimes\mathbf{Q}_{22}^{(k)}\right)\mathbf{e}(t-kh) \\ & + 2\mathbf{e}^{\mathrm{T}}(t-(k-1)h)\left(\mathbf{U}\otimes\mathbf{Q}_{23}^{(k)}\right)\mathbf{F}(\mathbf{e}(t-(k-1)h)) \\ & - 2\mathbf{e}^{\mathrm{T}}(t-kh)\left(\mathbf{U}\otimes\mathbf{Q}_{23}^{(k)}\right)\mathbf{F}(\mathbf{e}(t-kh)) \\ & + \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-(k-1)h))\left(\mathbf{U}\otimes\mathbf{Q}_{33}^{(k)}\right)\mathbf{F}(\mathbf{e}(t-(k-1)h)) \\ & - \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-kh))\left(\mathbf{U}\otimes\mathbf{Q}_{33}^{(k)}\right)\mathbf{F}(\mathbf{e}(t-kh)) \Big\}\end{aligned} \tag{9}$$

With reformulating (5) as:

$$\begin{aligned}\dot{\mathbf{e}}(t) = & -(\mathbf{I}_N\otimes\mathbf{C})\mathbf{e}(t) + (\mathbf{I}_N\otimes\mathbf{A})\mathbf{F}(\mathbf{e}(t)) + (\mathbf{I}_N\otimes\mathbf{B})\mathbf{F}(\mathbf{e}(t-\tau)) \\ & + \left(\mathbf{G}^{(1)}\otimes\mathbf{\Gamma}_1\right)\mathbf{e}(t) + \left(\mathbf{G}^{(2)}\otimes\mathbf{\Gamma}_2\right)\mathbf{e}(t-\tau) \\ & + \left(\mathbf{G}^{(3)}\otimes\mathbf{\Gamma}_3\right)\sum_{k=1}^{m}\int_{t-kh}^{t-(k-1)h} \mathbf{e}(s)\mathrm{d}s,\end{aligned} \tag{10}$$

and substituting in (9) and considering Lemma 3, (9) can be written as the following:

$$\begin{aligned}\dot{V}_1(t) = \sum_{k=1}^{m} \Bigg\{ \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \Bigg[& 2h\mathbf{e}_{ij}^{\mathrm{T}}(t)\Big[-\mathbf{Q}_{11}^{(k)}\mathbf{C}\mathbf{e}_{ij}(t) + \mathbf{Q}_{11}^{(k)}\mathbf{A}\mathbf{f}_{ij}(\mathbf{e}(t)) + \mathbf{Q}_{11}^{(k)}\mathbf{B}\mathbf{f}_{ij}(\mathbf{e}(t-\tau)) - NG_{ij}^{(1)}\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_1\mathbf{e}_{ij}(t) \\ & - NG_{ij}^{(2)}\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_2\mathbf{e}_{ij}(t-\tau) - NG_{ij}^{(3)}\mathbf{Q}_{11}^{(k)}\mathbf{\Gamma}_3\int_{t-kh}^{t-(k-1)h}\mathbf{e}_{ij}(s)\mathrm{d}s\Big] + 2\dot{\mathbf{e}}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{12}^{(k)}\int_{t-kh}^{t-(k-1)h}\mathbf{e}_{ij}(s)\mathrm{d}s \\ & + 2\mathbf{e}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{12}^{(k)}\mathbf{e}_{ij}(t-(k-1)h) - 2\mathbf{e}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{12}^{(k)}\mathbf{e}_{ij}(t-kh) + 2\dot{\mathbf{e}}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{13}^{(k)}\int_{t-kh}^{t-(k-1)h}\mathbf{f}_{ij}(\mathbf{e}(s))\mathrm{d}s \\ & + 2\mathbf{e}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{13}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-(k-1)h)) - 2\mathbf{e}_{ij}^{\mathrm{T}}(t)\mathbf{Q}_{13}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-kh)) + \mathbf{e}_{ij}^{\mathrm{T}}(t-(k-1)h)\mathbf{Q}_{22}^{(k)}\mathbf{e}_{ij}(t-(k-1)h) \\ & - \mathbf{e}_{ij}^{\mathrm{T}}(t-kh)\mathbf{Q}_{22}^{(k)}\mathbf{e}_{ij}(t-kh) + 2\mathbf{e}_{ij}^{\mathrm{T}}(t-(k-1)h)\mathbf{Q}_{23}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-(k-1)h)) \\ & - 2\mathbf{e}_{ij}^{\mathrm{T}}(t-kh)\mathbf{Q}_{23}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-kh)) + \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-(k-1)h))\mathbf{Q}_{33}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-(k-1)h)) \\ & - \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-kh))\mathbf{Q}_{33}^{(k)}\mathbf{f}_{ij}(\mathbf{e}(t-kh))\Bigg]\Bigg\}.\end{aligned} \tag{11}$$

The second term of (8) becomes:

$$\dot{V}_2(t) = 2\begin{bmatrix} \mathbf{e}(t) \\ \Upsilon \\ \Theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{Z} & \mathbf{U}\otimes\mathbf{O}^{(1)} & \mathbf{U}\otimes\mathbf{O}^{(2)} \\ * & \mathbf{U}\otimes\mathbf{W}^{(1)} & \mathbf{U}\otimes\mathbf{W}^{(2)} \\ * & * & \mathbf{U}\otimes\mathbf{W}^{(3)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}}(t) \\ \dot{\Upsilon} \\ \dot{\Theta} \end{bmatrix} \tag{12}$$

where,

$$\dot{\Upsilon} = \left[\mathbf{e}^{\mathrm{T}}(t) - \mathbf{e}^{\mathrm{T}}(t-h) \quad \mathbf{e}^{\mathrm{T}}(t-h) - \mathbf{e}^{\mathrm{T}}(t-2h) \quad \cdots \quad \mathbf{e}^{\mathrm{T}}(t-(m-1)h) - \mathbf{e}^{\mathrm{T}}(t-\tau)\right]^{\mathrm{T}},$$
$$\dot{\Theta} = \left[\mathbf{F}^{\mathrm{T}}(\mathbf{e}(t)) - \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-h)) \quad \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-h)) - \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-2h)) \quad \cdots \quad \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-(m-1)h)) - \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-\tau))\right]^{\mathrm{T}}.$$

According to Lemma 3 and considering (10), (12) can be written as the following:

$$\dot{V}_2(t) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[2\begin{bmatrix} \mathbf{e}_{ij}(t) \\ \Upsilon_{ij} \\ \Theta_{ij} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{Z} & \mathbf{O}^{(1)} & \mathbf{O}^{(2)} \\ * & \mathbf{W}^{(1)} & \mathbf{W}^{(2)} \\ * & * & \mathbf{W}^{(3)} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{e}}_{ij}(t) \\ \dot{\Upsilon}_{ij} \\ \dot{\Theta}_{ij} \end{bmatrix}\right]$$

where,

$$\Upsilon_{ij} = \left[\int_{t-h}^{t}\mathbf{e}_{ij}^{\mathrm{T}}(s)\mathrm{d}s \quad \int_{t-2h}^{t-h}\mathbf{e}_{ij}^{\mathrm{T}}(s)\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h}\mathbf{e}_{ij}^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}},$$
$$\Theta_{ij} = \left[\int_{t-h}^{t}\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \int_{t-2h}^{t-h}\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h}\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s\right]^{\mathrm{T}},$$
$$\dot{\Upsilon}_{ij} = \left[\mathbf{e}_{ij}^{\mathrm{T}}(t) - \mathbf{e}_{ij}^{\mathrm{T}}(t-h) \quad \mathbf{e}_{ij}^{\mathrm{T}}(t-h) - \mathbf{e}_{ij}^{\mathrm{T}}(t-2h) \quad \cdots \quad \mathbf{e}_{ij}^{\mathrm{T}}(t-(m-1)h) - \mathbf{e}_{ij}^{\mathrm{T}}(t-\tau)\right]^{\mathrm{T}},$$
$$\dot{\Theta}_{ij} = \left[\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t)) - \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-h)) \quad \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-h)) - \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-2h)) \quad \cdots \quad \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-(m-1)h)) - \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-\tau))\right]^{\mathrm{T}}.$$

Taking the derivative of $V_3(t)$ with respect to t yields:

$$\begin{aligned}\dot{V}_3(t) = \sum_{k=1}^{m}\Bigg\{ & h^2\begin{bmatrix} \mathbf{e}(t) \\ \dot{\mathbf{e}}(t) \\ \mathbf{F}(\mathbf{e}(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{R}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{R}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{R}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \dot{\mathbf{e}}(t) \\ \mathbf{F}(\mathbf{e}(t)) \end{bmatrix} \\ & -h\int_{t-kh}^{t-(k-1)h} \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{R}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{R}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{R}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}\mathrm{d}s\Bigg\}\end{aligned} \tag{13}$$

According to Lemma 1, (13) can be written as the following:

$$\begin{aligned}\dot{V}_3(t) \le \sum_{k=1}^{m}\Bigg\{ & h^2\begin{bmatrix} \mathbf{e}(t) \\ \dot{\mathbf{e}}(t) \\ \mathbf{F}(\mathbf{e}(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{R}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{R}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{R}_{33}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{e}(t) \\ \dot{\mathbf{e}}(t) \\ \mathbf{F}(\mathbf{e}(t)) \end{bmatrix} \\ & -\left(\int_{t-kh}^{t-(k-1)h} \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}\mathrm{d}s\right)^{\mathrm{T}} \begin{bmatrix} \mathbf{U}\otimes\mathbf{R}_{11}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{12}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{13}^{(k)} \\ * & \mathbf{U}\otimes\mathbf{R}_{22}^{(k)} & \mathbf{U}\otimes\mathbf{R}_{23}^{(k)} \\ * & * & \mathbf{U}\otimes\mathbf{R}_{33}^{(k)} \end{bmatrix} \left(\int_{t-kh}^{t-(k-1)h} \begin{bmatrix} \mathbf{e}(s) \\ \dot{\mathbf{e}}(s) \\ \mathbf{F}(\mathbf{e}(s)) \end{bmatrix}\mathrm{d}s\right)\Bigg\}\end{aligned} \tag{14}$$

According to Lemma 3, (14) can be written as the following:

$$\dot{V}_3(t) \leq \sum_{k=1}^{m}\left\{\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[h^2\begin{bmatrix}\mathbf{e}_{ij}(t)\\ \dot{\mathbf{e}}_{ij}(t)\\ \mathbf{f}_{ij}(\mathbf{e}_i(t))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{R}_{11}^{(k)} & \mathbf{R}_{12}^{(k)} & \mathbf{R}_{13}^{(k)}\\ * & \mathbf{R}_{22}^{(k)} & \mathbf{R}_{23}^{(k)}\\ * & * & \mathbf{R}_{33}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{ij}(t)\\ \dot{\mathbf{e}}_{ij}(t)\\ \mathbf{f}_{ij}(\mathbf{e}_i(t))\end{bmatrix}\right.\right.$$
$$\left.\left.-\begin{bmatrix}\int_{t-kh}^{t-(k-1)h}\mathbf{e}_{ij}(s)\mathrm{d}s\\ \int_{t-kh}^{t-(k-1)h}\dot{\mathbf{e}}_{ij}(s)\mathrm{d}s\\ \int_{t-kh}^{t-(k-1)h}\mathbf{f}_{ij}(\mathbf{e}(s))\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{R}_{11}^{(k)} & \mathbf{R}_{12}^{(k)} & \mathbf{R}_{13}^{(k)}\\ * & \mathbf{R}_{22}^{(k)} & \mathbf{R}_{23}^{(k)}\\ * & * & \mathbf{R}_{33}^{(k)}\end{bmatrix}\begin{bmatrix}\int_{t-kh}^{t-(k-1)h}\mathbf{e}_{ij}(s)\mathrm{d}s\\ \int_{t-kh}^{t-(k-1)h}\dot{\mathbf{e}}_{ij}(s)\mathrm{d}s\\ \int_{t-kh}^{t-(k-1)h}\mathbf{f}_{ij}(\mathbf{e}(s))\mathrm{d}s\end{bmatrix}\right]\right\} \tag{15}$$

According to Lemma 2 and Assumption 2, for any positive diagonal matrices $\mathbf{J}$ and $\mathbf{L}$, one has:

$$\boldsymbol{\theta}^{\mathrm{T}}(t)\begin{bmatrix}-\mathbf{J}\Delta_1 & \mathbf{J}\Delta_2\\ * & -\mathbf{J}\end{bmatrix}\boldsymbol{\theta}(t) + \boldsymbol{\theta}^{\mathrm{T}}(t-\tau)\begin{bmatrix}-\mathbf{L}\Delta_1 & \mathbf{L}\Delta_2\\ * & -\mathbf{L}\end{bmatrix}\boldsymbol{\theta}(t-\tau) \geq 0, \tag{16}$$

where $\boldsymbol{\theta}(t) = \begin{bmatrix}\mathbf{e}_{ij}^{\mathrm{T}}(t) & \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t))\end{bmatrix}^{\mathrm{T}}$.

From Eq. (5), the following equation holds for any matrices $\mathbf{T}_q \in \mathbb{R}^{n\times n}$ and $\mathbf{H}_k \in \mathbb{R}^{n\times n}$, $(q = 1,\ldots,5,\ k = 1,\ldots,m)$:

$$\begin{aligned}0 = 2\Big\{&\dot{\mathbf{e}}^{\mathrm{T}}(t)(\mathbf{I}_N\otimes\mathbf{T}_1) + \mathbf{e}^{\mathrm{T}}(t)(\mathbf{I}_N\otimes\mathbf{T}_2) + \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t))(\mathbf{I}_N\otimes\mathbf{T}_3)\\ &+\mathbf{e}^{\mathrm{T}}(t-\tau)(\mathbf{I}_N\otimes\mathbf{T}_4) + \mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-\tau))(\mathbf{I}_N\otimes\mathbf{T}_5)\\ &+\sum_{k=1}^{m}\left(\int_{t-kh}^{t-(k-1)h}\mathbf{F}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s(\mathbf{I}_N\otimes\mathbf{H}_k)\right)\Big\}\\ &[-\dot{\mathbf{e}}(t) - (\mathbf{I}_N\otimes\mathbf{C})\mathbf{e}(t) + (\mathbf{I}_N\otimes\mathbf{A})\mathbf{F}(\mathbf{e}(t)) + (\mathbf{I}_N\otimes\mathbf{B})\mathbf{F}(\mathbf{e}(t-\tau))\\ &+\left(\mathbf{G}^{(1)}\otimes\boldsymbol{\Gamma}_1\right)\mathbf{e}(t) + \left(\mathbf{G}^{(2)}\otimes\boldsymbol{\Gamma}_2\right)\mathbf{e}(t-\tau)\\ &+\left(\mathbf{G}^{(3)}\otimes\boldsymbol{\Gamma}_3\right)\sum_{k=1}^{m}\int_{t-kh}^{t-(k-1)h}\mathbf{e}(s)\mathrm{d}s\Big].\end{aligned} \tag{17}$$

In addition, based on Leibniz–Newton formula, the following equation holds for any matrices $\mathbf{P}_q \in \mathbb{R}^{n\times n}$ and $\mathbf{Y}_k \in \mathbb{R}^{n\times n}$, $(q = 1,\ldots,5,\ k = 1,\ldots,m)$:

$$\begin{aligned}0 = 2\Big\{&\dot{\mathbf{e}}^{\mathrm{T}}(t)(\mathbf{I}_N\otimes\mathbf{P}_1) + \mathbf{e}^{\mathrm{T}}(t)(\mathbf{I}_N\otimes\mathbf{P}_2)\\ &+\mathbf{F}^{\mathrm{T}}(\mathbf{e}(t))(\mathbf{I}_N\otimes\mathbf{P}_3) + \mathbf{e}^{\mathrm{T}}(t-\tau)(\mathbf{I}_N\otimes\mathbf{P}_4)\\ &+\mathbf{F}^{\mathrm{T}}(\mathbf{e}(t-\tau))(\mathbf{I}_N\otimes\mathbf{P}_5)\\ &+\sum_{k=1}^{m}\left(\int_{t-kh}^{t-(k-1)h}\dot{\mathbf{e}}^{T}(s)\mathrm{d}s(\mathbf{I}_N\otimes\mathbf{Y}_k)\right)\Big\}\\ &\left[\mathbf{e}(t) - \mathbf{e}(t-\tau) - \sum_{k=1}^{m}\left(\int_{t-kh}^{t-(k-1)h}\dot{\mathbf{e}}(s)\mathrm{d}s\right)\right].\end{aligned} \tag{18}$$

Considering (9)–(18), it is straightforward to show that:

$$\dot{V}(t) \leq \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[\boldsymbol{\xi}_{ij}^{\mathrm{T}}(t)\boldsymbol{\Psi}_{ij}\boldsymbol{\xi}_{ij}(t)\right], \tag{19}$$

where $\boldsymbol{\Psi}_{ij}$ is defined in (7) and

$$\begin{aligned}\boldsymbol{\xi}_{ij}(t) = \Big[&\mathbf{e}_{ij}^{\mathrm{T}}(t) \quad \mathbf{e}_{ij}^{\mathrm{T}}(t-h) \quad \cdots \quad \mathbf{e}_{ij}^{\mathrm{T}}(t-(m-1)h) \quad \mathbf{e}_{ij}^{\mathrm{T}}(t-\tau) \quad \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t))\\ &\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-h)) \quad \cdots \quad \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-(m-1)h)) \quad \mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(t-\tau)) \quad \int_{t-h}^{t-2h}\mathbf{e}_{ij}^{\mathrm{T}}(s)\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h}\mathbf{e}_{ij}^{\mathrm{T}}(s)\mathrm{d}s\\ &\int_{t-h}^{t-2h}\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \cdots \quad \int_{t-\tau}^{t-(m-1)h}\mathbf{f}_{ij}^{\mathrm{T}}(\mathbf{e}(s))\mathrm{d}s \quad \dot{\mathbf{e}}_{ij}^{\mathrm{T}}(t) \quad \int_{t-h}^{t-2h}\dot{\mathbf{e}}_{ij}^{\mathrm{T}}(s)\mathrm{d}s \quad \cdots \quad \int_{t-h}^{t-2h}\dot{\mathbf{e}}_{ij}^{\mathrm{T}}(s)\mathrm{d}s\Big]^{\mathrm{T}}.\end{aligned}$$

If $\boldsymbol{\Psi}_{ij} < 0$ for $\forall 1 \le i < j \le N$, then $\dot{V}(t) < 0$. From Definition 1, this implies that the system (1) has a global synchronization which leads in completing the proof. □

4 Illustrative example

Example Consider the chaotic cellular neural network with the following equations which is presented in [36]:

$$\dot{\mathbf{x}}(t) = -\mathbf{C}\mathbf{x}(t) + \mathbf{A}\mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{f}(\mathbf{x}(t-\tau)), \tag{20}$$

where $\mathbf{x}(t) = \left[x_1^{\mathrm{T}}(t) \quad x_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is the state vector of the network, $f(x_i(t)) = 0.5(|x_i + 1| - |x_i - 1|)$ is the activation function, and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 1 + \frac{\pi}{4} & 20 \\ 0.1 & 1 + \frac{\pi}{4} \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -\frac{1.3\pi\sqrt{2}}{4} & 0.1 \\ 0.1 & -\frac{1.3\pi\sqrt{2}}{4} \end{bmatrix}.$$

For $\tau = 0.97$ and initial condition as $\mathbf{x}_0(t) = [10, -15]^{\mathrm{T}}$, the chaotic behavior of (20) is shown in Fig. 1.

Let couple three number of identical neural networks (20) as (1) with the following coupling parameters:

$$\mathbf{G}^{(1)} = \mathbf{G}^{(2)} = \mathbf{G}^{(3)} = \begin{bmatrix} -8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & -8 \end{bmatrix},$$

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \ \boldsymbol{\Gamma}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{\Gamma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

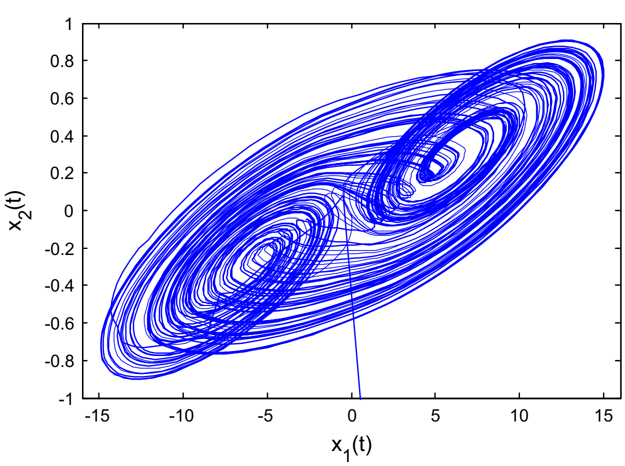

Fig. 1 Chaotic trajectory of (20)

Table 1 Maximum allowable time delay for different methods

Methods	Maximum allowable time delay (τ)	Computation time (s)
[36]	0.699	0.046
[21] ($h_1 = 0,\ \mu = 0$)	1.050	3.822
Theorem 1 ($m = 1$)	2.028	0.764
Theorem 1 ($m = 2$)	5.568	4.773
Theorem 1 ($m = 3$)	8.586	21.902

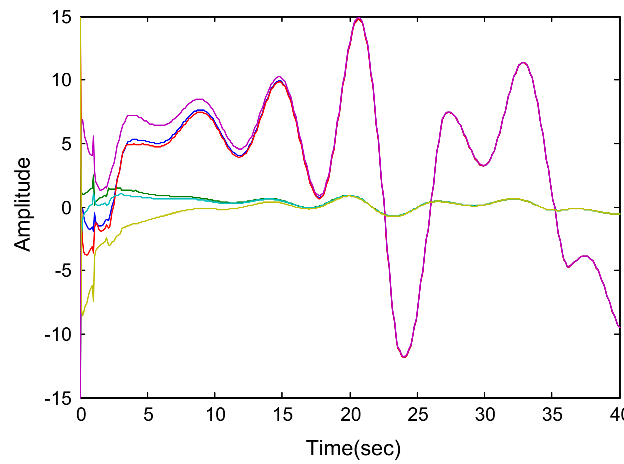

Fig. 2 State trajectories of the coupled neural networks: $\mathbf{x}_i(t);\ i = 1, 2, 3$

Maximum allowable time delay for different methods which guarantees the synchronization between the neural networks is shown in Table 1. It is obvious that the theorem introduced in this paper is less conservative than those proposed in [21, 36]. Note that the results for Theorem 1 can be more considerable by increasing the segmentation number (m). As shown in Table 1, the main disadvantage of segmentation method is the computation time of solving the LMIs which are increasing rapidly.

With different initial conditions for the neural networks, the state trajectories of them are shown in Fig. 2. The synchronization errors are shown in Fig. 3, where $\mathbf{e}_j(t) = \left(\mathbf{x}_{1j}(t) - \mathbf{x}_{ij}(t)\right)$, $i = 2, 3;\ j = 1, 2$.

5 Conclusion

This paper studied the problem of global synchronization of coupled neural networks with hybrid coupling. Based on a new augmented Lyapunov–Krasovskii functional and the

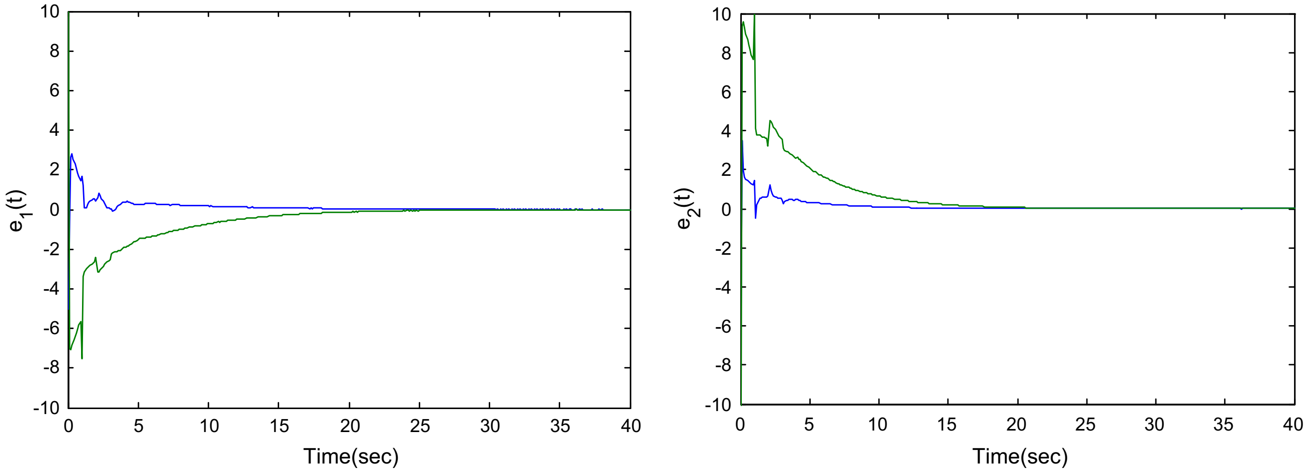

Fig. 3 Synchronization errors for the networks: $\mathbf{e}_j(t),\ j = 1, 2$

idea of M-segmentation of delay length, a less conservative delay-dependent criterion was obtained and expressed in the form of linear matrix inequalities. As an example, a typical chaotic cellular neural network was utilized to show that the proposed method is less conservative than the mentioned common methods described in this paper.

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